Syllabus

1. Introduction

2. Fluids

1. General Characteristics
2. Dispersions
3. Thermodynamics
4. **Transport Phenomena**
5. Solutions
6. Surface Tension
7. Electrical Properties
8. Optical Properties
9. Biological Fluids

Physics of Microfluidic Systems

1. Navier-Stokes Equation
2. Laminar and Turbulent Flow
3. Fluid Dynamics
4. Fluid Networks
5. Transport of Heat
6. Interfacial Surface Tension
7. Electrokinetics

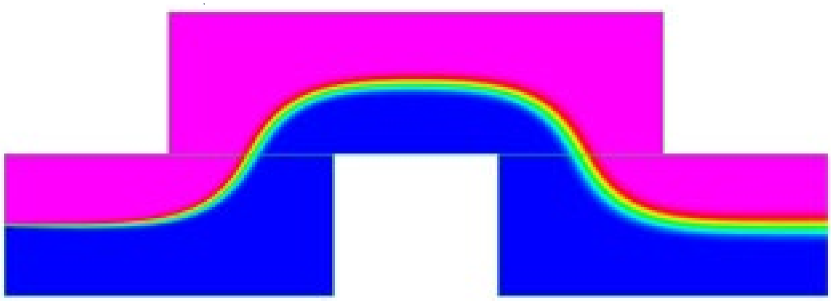
2.4. Transport Phenomena

2.4.1. Diffusion

2.4.2. Viscosity

2.4.3. Transport of Heat

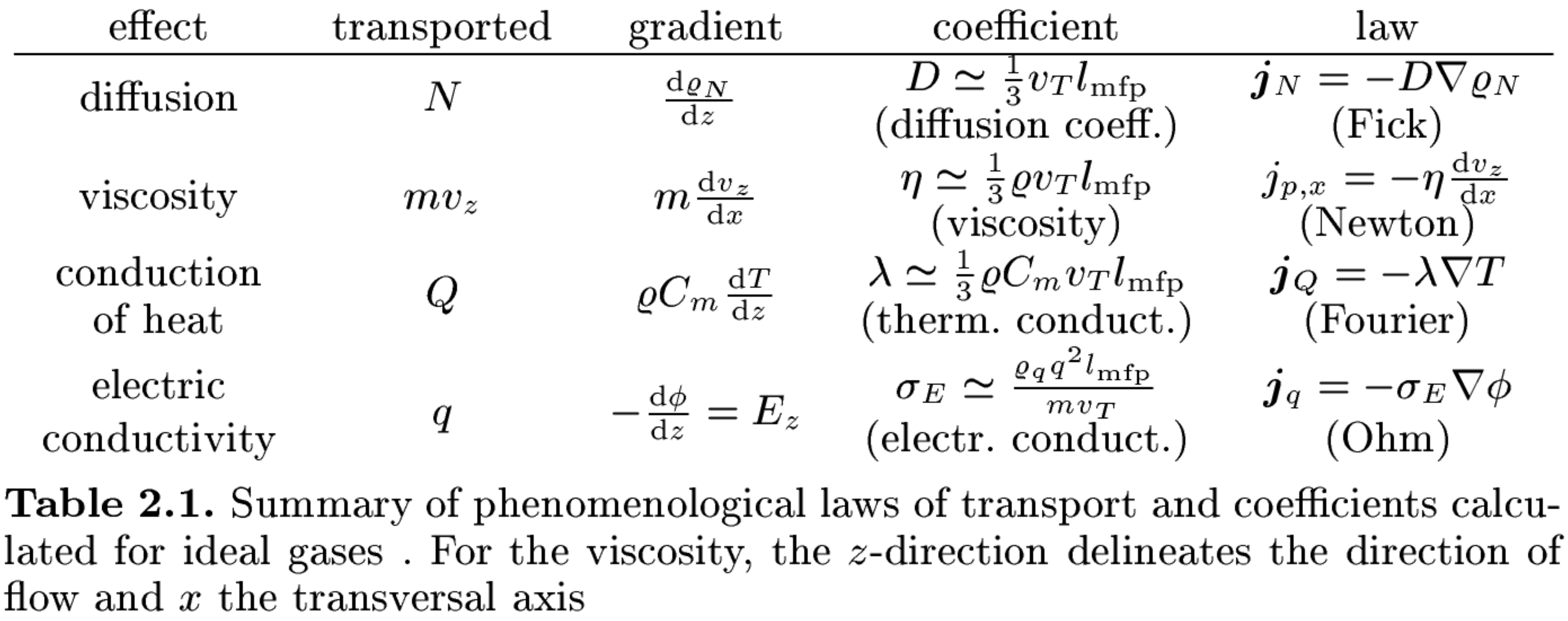
2.4.4. Characteristic Numbers



2.4 Summary of Phenomenological Laws

- Transport processes
  - Arbitrary thermal motion on molecular level
  - Driven by gradients (inhomogeneities)
  - Mostly linear relationship

    spread of distribution = coefficient × gradient



effect	transported	gradient	coefficient	law
diffusion	N	$\frac{\mathrm{d}\varrho_N}{\mathrm{d}z}$	$D \simeq \frac{1}{3} v_T l_{\mathrm{mfp}}$ (diffusion coeff.)	$\boldsymbol{j}_N = -D\nabla\varrho_N$ (Fick)
viscosity	mv_z	$m\frac{\mathrm{d}v_z}{\mathrm{d}x}$	$\eta \simeq \frac{1}{3}\varrho v_T l_{\mathrm{mfp}}$ (viscosity)	$j_{p,x} = -\eta\frac{\mathrm{d}v_z}{\mathrm{d}x}$ (Newton)
conduction of heat	Q	$\varrho C_m \frac{\mathrm{d}T}{\mathrm{d}z}$	$\lambda \simeq \frac{1}{3}\varrho C_m v_T l_{\mathrm{mfp}}$ (therm. conduct.)	$\boldsymbol{j}_Q = -\lambda\nabla T$ (Fourier)
electric conductivity	q	$-\frac{\mathrm{d}\phi}{\mathrm{d}z} = E_z$	$\sigma_E \simeq \frac{\varrho_q q^2 l_{\mathrm{mfp}}}{m v_T}$ (electr. conduct.)	$\boldsymbol{j}_q = -\sigma_E\nabla\phi$ (Ohm)

Table 2.1. Summary of phenomenological laws of transport and coefficients calculated for ideal gases . For the viscosity, the z-direction delineates the direction of flow and x the transversal axis

2.4.1 Brownian Motion

- Discovered in 1827 by Scottish botanist Robert Brown
- Random thermal motion of pollen under microscope
- Closely related to thermal velocity

$$v_T = \sqrt{\bar{v}^2} = \sqrt{\frac{3k_B \boxed{T}}{m}}$$

- Mesoscopic particle
  - $r_0 = 1\ \mu\text{m}$
  - $m = 10^{-15}$ kg
  - $T = 293$ K
  - $v_T \approx 3$ mm s^{-1}

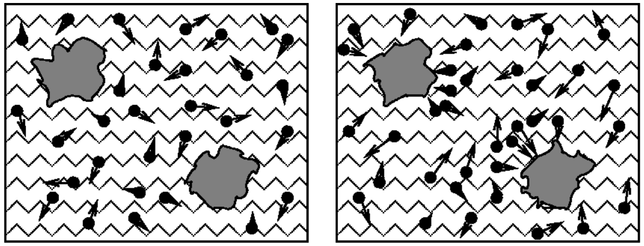
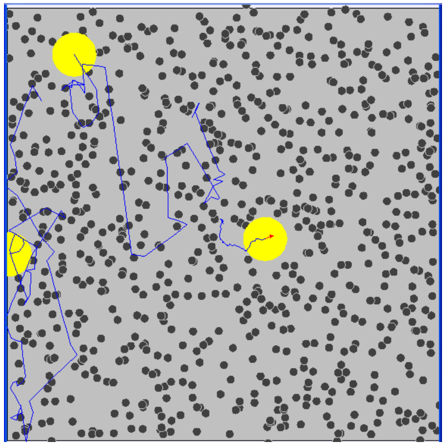


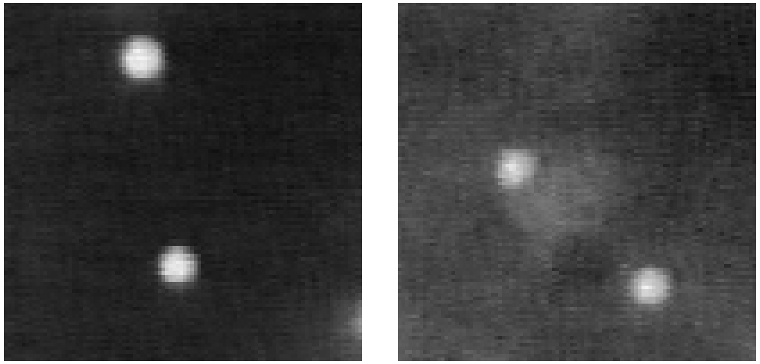
Fig. 2.10. Brownian motion resulting from random molecular pressure oscillations leads to wobbling motion of mesoscopic particles that can be observed under a microscope

Width of particle distribution

$$\bar{z}^2 = 3Dt = \frac{k_B \boxed{T}}{2\pi \boxed{\eta} \boxed{r_0}} \boxed{t}$$



2.4.1 Brownian Motion



- Fluorescent particles
- 4 seconds of data
- 2 µm in diameter
- Left picture
  - Particles moving in pure water
- Right picture
  - Particles moving in concentrated viscoelastic solution of DNA

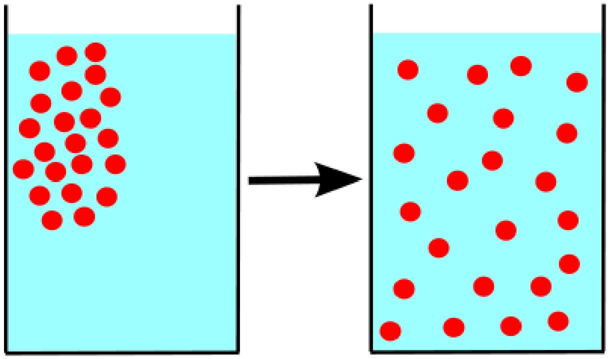
From http://www.deas.harvard.edu/projects/weitzlab/research/brownian.html

2.4.1. Diffusion

- Counteracts nonuniform particle densities
- Thermal „Brownian“ motion
- Process underlying all other transport phenomena in fluids

- Fick‘s first law

$$\boxed{j_N = -D \nabla \varrho_N}$$



- ➢ “Current” density j_N antiparallel to gradient
- ➢ System seeks homogeneity
- ➢ Diffusion coefficient D $[\mathrm{m}^2\ \mathrm{s}^{-1}]$

2.4.1. Diffusion

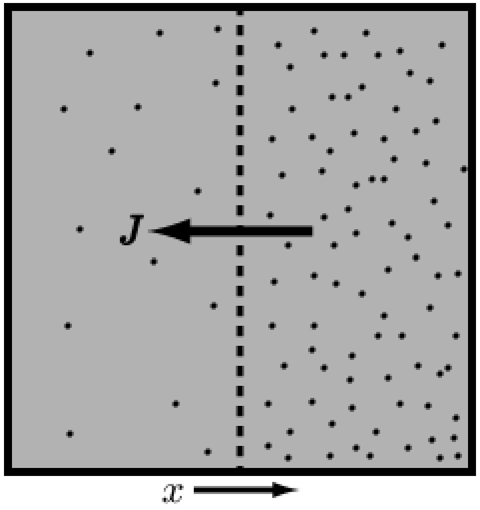
- Fick‘s second law
    - Time domain
    - Fixed location

$$\boxed{\frac{\partial \varrho_N}{\partial t} = D \triangle \varrho_N}$$

- Combining Fick‘s first law and equation of continuity

$$\boxed{j_N = -D \nabla \varrho_N}$$

$$\boxed{\frac{\partial \varrho_N}{\partial t} = -\nabla \cdot j_N}$$



- Laplace equation
    - Stationary conditions

$$\boxed{\triangle \varrho_N = 0}$$

2.4.1 Molecular Picture

optional

- Derivation of diffusion constant *D*
- Flow through surface z_1
  - Between two planes with diverging particle densities
  - Distance = mean free path l_{mfp}
  - Particles reach plane without collisions (statistically)

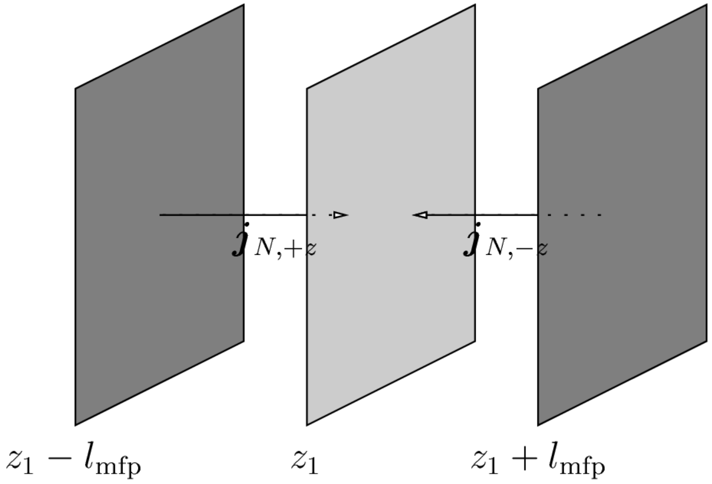


Fig. 2.7. Net flow through surface located at $z = z_1$

$$j_{N,z} = j_{N,+z} - j_{N,-z} = \frac{1}{2}[\varrho_N(z - l_{\mathrm{mfp}}) - \varrho_N(z + l_{\mathrm{mfp}})]\bar{v}_z$$

$$j_{N,z} \simeq -\boxed{\frac{\mathrm{d}\varrho_N}{\mathrm{d}z}}\boxed{\bar{v}_z l_{\mathrm{mfp},z}} \qquad j_N = -\boxed{D}\boxed{\nabla \varrho_N}$$

  - Linear term of Taylor expansion

2.4.1 Diffusion Coefficient - Calculation

- Assumption: uniformly distributed directions of all vectors v

$$D = < \bar{v}_z l_{\mathrm{mfp},z} > = v_T l_{\mathrm{mfp}} \frac{1}{2\pi} \int_0^{\pi/2} \cos^2 \Theta \, 2\pi \sin \Theta \, \mathrm{d}\Theta$$

average over ensemble

$$\boxed{D = \frac{1}{3} v_T l_{\mathrm{mfp}}}$$

Mean thermal velocity v_T:

- Temperature
- Mass

Mean free path l_{mfp}:

- Density / pressure

solute	solvent	D / $10^{-9}\,\mathrm{m^2\,s^{-1}}$
I_2	C_6H_{12}	4.05
I_2	CCl_4	3.42
I_2	C_6H_6	2.13
N_2	CCl_4	3.42
O_2	CCl_4	3.82
Ar_2	CCl_4	3.63
H_2O	H_2O (self diffusion)	2.62
dextrose	H_2O	0.67
H^+	H_2O	9.31
Li^+	H_2O	1.03
Na^+	H_2O	1.96
Cl^-	H_2O	2.03
Br^-	H_2O	2.08
I^-	H_2O	2.05

Table 2.1. Diffusion coefficients D of molecules and ions in various solvents

2.4.1 Examples for Diffusion Processes

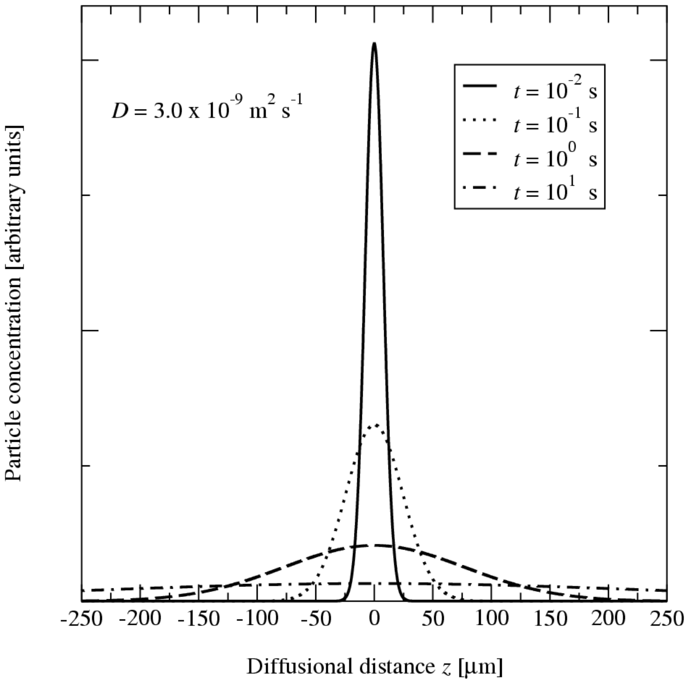
- Bilateral diffusion from 2D layer
  - Within pure solvent
  - Initial conditions

$$\varrho_N(t=0, z=0) = \varrho_{N,0}$$

$$\varrho_N(t=0, z\neq 0) = 0$$

- Solution by „educated guess“

$$\varrho_N(z,t) = \frac{\varrho_{N,0}}{\sqrt{\pi D t}} \mathrm{e}^{-\frac{\boxed{z^2}}{4\boxed{Dt}}}$$



2.4.1 Examples for Diffusive Processes

- Example
  - Diffusion through (suddenly) permeable diaphragm
  - Initial conditions

$$\varrho_N(t=0, z<0) = \varrho_{N,0}$$

$$\varrho_N(t=0, z>0) = 0$$

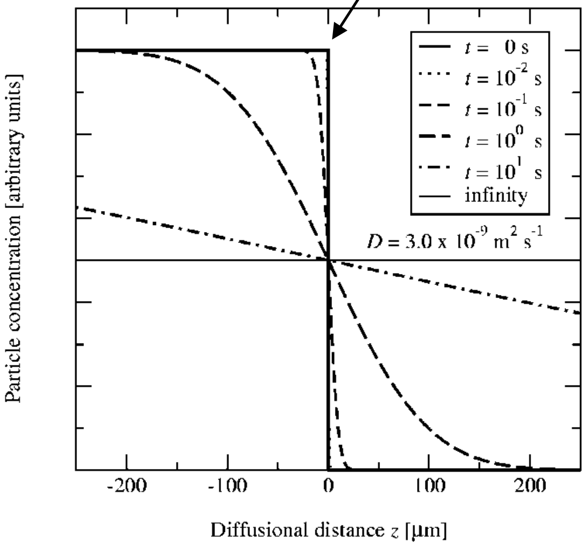
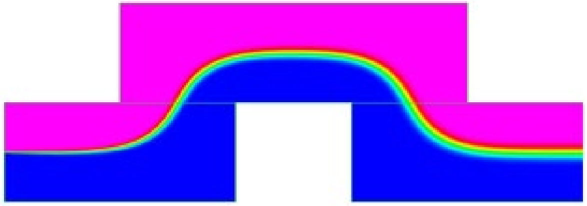


Fig. 2.9. Diffusion of molecules through a permeable wall at $z = 0$ which are initially located at $z < 0$. $D = 3.0 \times 10^{-9}\,\mathrm{m^2\,s^{-1}}$ has been assumed. With increasing time, a uniform distribution builds out



2.4.1 Time and Length Scales

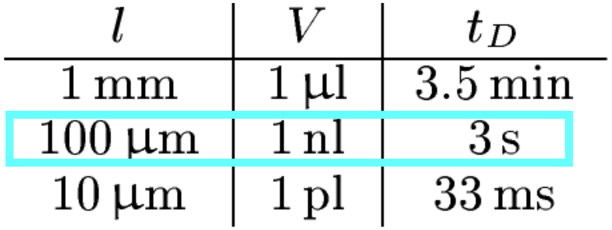
- Distance

$$l_D = \sqrt{2D\boxed{t}}$$

- Time

("rules of thumb")

$$t_D = \frac{\boxed{l^2}}{2D}$$



l	V	t_D
1 mm	1 µl	3.5 min
100 µm	1 nl	3 s
10 µm	1 pl	33 ms

Table 2.8. Typical diffusion time t for lengths l and corresponding volumes V for $D = 3 \times 10^{-9}\,\mathrm{m}^2\,\mathrm{s}^{-1}$

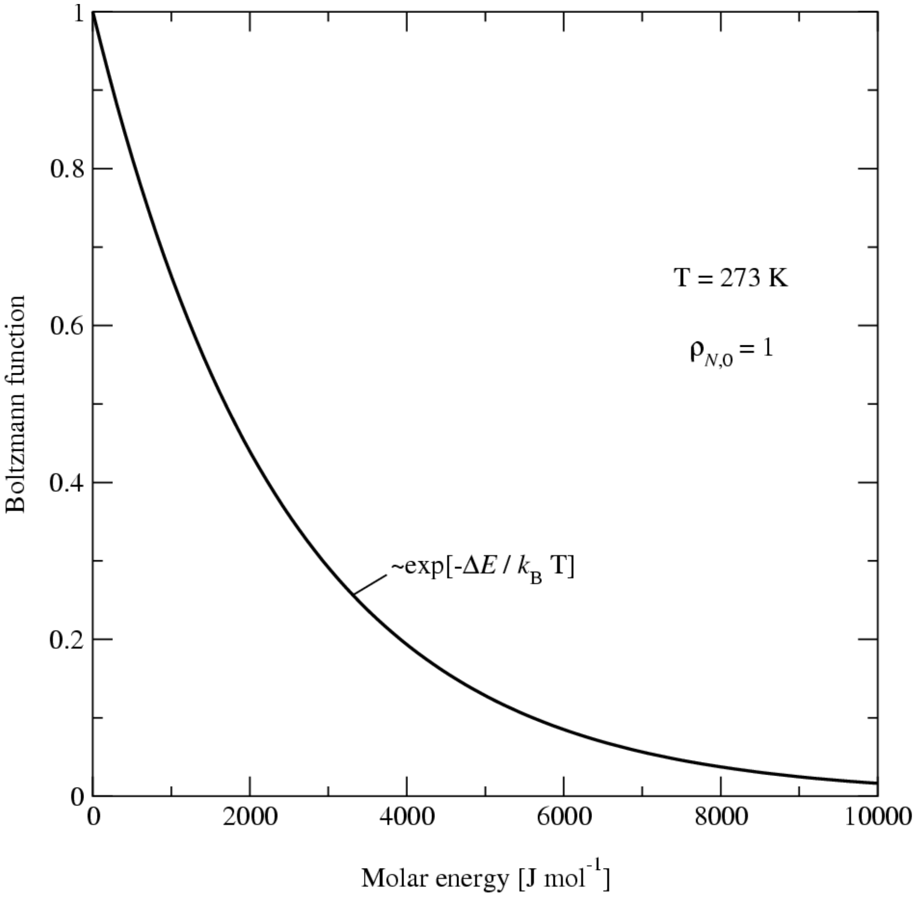
2.4.1 Boltzmann Distribution – Energy Space

- System in thermal equilibrium at T
- Each particle with certain kinetic energy $E = \frac{1}{2} m v^2$
- Competition
  - Potential gradient
    - Particles accumulate in potential minimum
  - Counteracting diffusive current

$$j_N = -D \nabla \varrho_N$$

- Boltzmann distribution
  - Ratio of particle densities ρ_N separated by ΔE

$$\frac{\varrho_{N,1}}{\varrho_{N,0}} = \mathrm{e}^{-\frac{\boxed{\Delta E}}{k_\mathrm{B} \boxed{T}}}$$



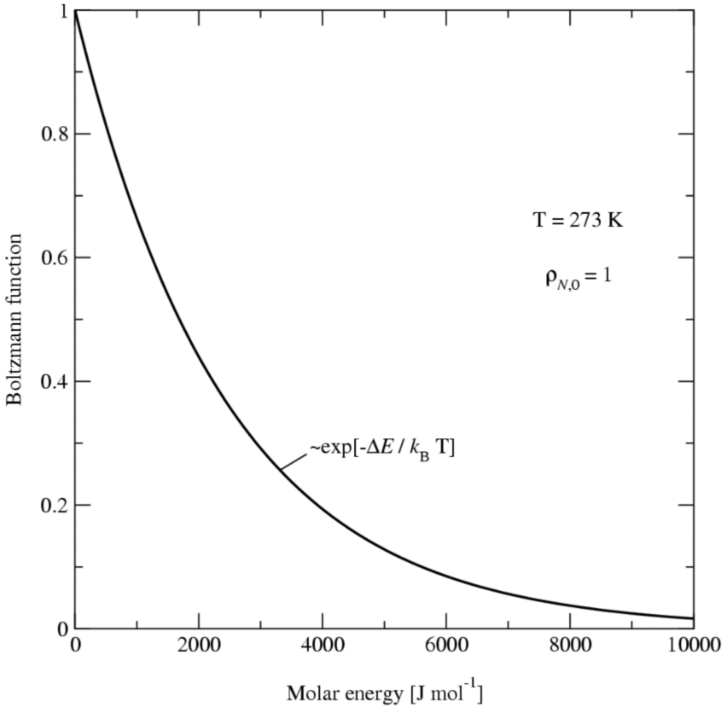
Boltzmann function at $T = 273$ K calibrated to $\rho_{N,0} = 1$

2.4.1 Boltzmann Distribution – Energy Space

- Formula

$$f(E)\mathrm{d}E = \frac{2}{\sqrt{\pi}}(k_\mathrm{B}T)^{-3/2}\sqrt{E}\mathrm{e}^{-E/k_\mathrm{B}T}\mathrm{d}E$$

- Tail fraction $E > E_0$
  - Helps determining which fraction of particles capable of „jumping“ over certain potential energy barrier E_0
  - Analytical expression for fraction with kinetic energy above E



$$\boxed{\text{tail fraction} = \frac{\int_{E_0}^{\infty} f(E)\mathrm{d}E}{\int_{0}^{\infty} f(E)\mathrm{d}E} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{E_0}{k_\mathrm{B}T}}\mathrm{e}^{-E_0/k_\mathrm{B}T}}$$

2.4.1 Maxwell Distribution – Velocity Space

- Fraction / probability of molecules in velocity interval $[v, v + dv]$

- Competition
  - ➢ Boltzmann decreases to with $E = \frac{1}{2}mv^2$
  - ➢ 3D-velocity space increases with v

- Normalization of integrand function $f(v)$

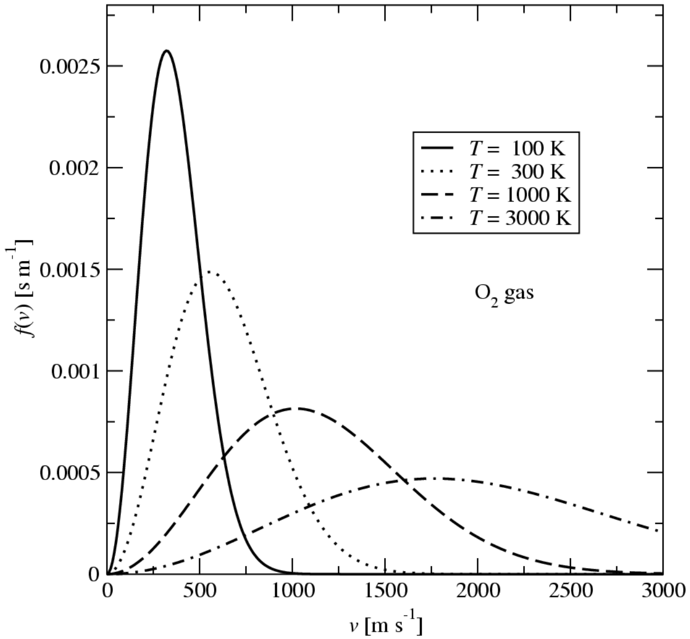


Fig. 2.12. Maxwellian function $f(v)$ for O_2 gas at different temperatures

$$\int_0^\infty f(v)\mathrm{d}v = 1$$

$$\boxed{f(v)\mathrm{d}v = \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_\mathrm{B}T}\right)^{3/2} v^2 \mathrm{e}^{-mv^2/2k_\mathrm{B}T}\mathrm{d}v}$$

2.4.1 Reaction Kinetics

optional

- Chemical Reactions
  - Binary collisions
  - Collision rate scales linearly with v_T and thus with $T^{1/2}$
  - Concentration of reaction partners *c* also influences speed of reaction
  - For single step reaction

$$x\mathrm{X} + y\mathrm{Y} + \ldots \longrightarrow \mathrm{products}$$

  - Reaction rate

$$\boxed{\frac{\mathrm{d}n}{\mathrm{d}t} = k_c' c^x(\mathrm{X}) c^y(\mathrm{Y}) \ldots}$$

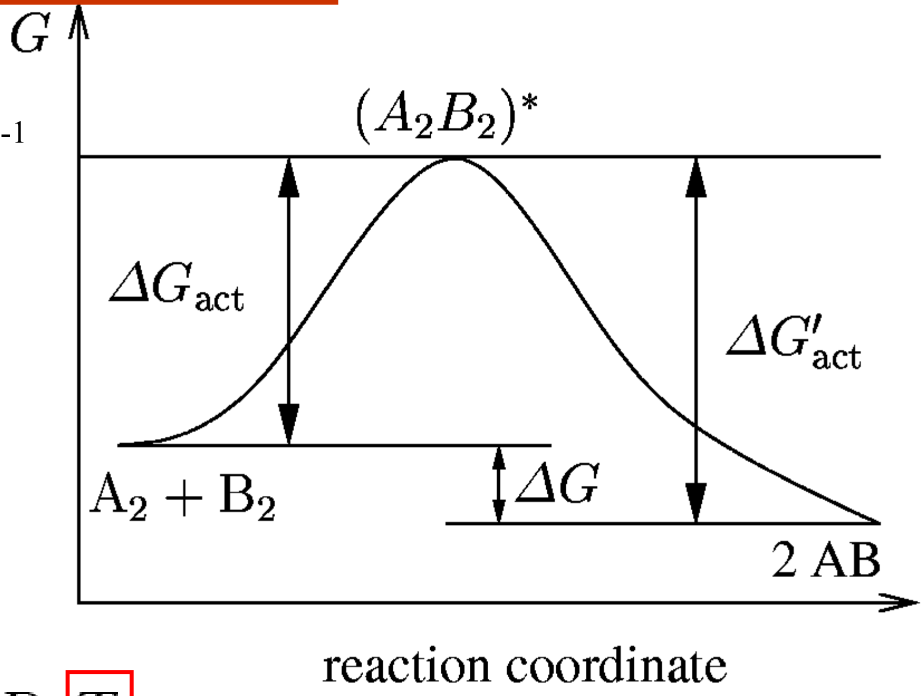
- Governed by stoichiometry $(x, y, \ldots)$
  - Concentrations $c(\mathrm{X}), c(\mathrm{Y}), \ldots$
  - Reaction velocity constant k_c'

2.4.1 Reaction Kinetics

optional

- Chemical reactions
  - Activation energy (Gibbs energy)
  - Simple model reaction

$$\boxed{\mathrm{A_2 + B_2 \rightleftharpoons (A_2B_2)^* \rightleftharpoons 2AB}}$$



  - For simplicity: $\Delta G = \Delta U = E_\text{act}$
  - Typically, $E_\text{act} = 60$ to $250\ \text{kJ mol}^{-1}$
  - $\Delta G = G_\text{act} - G'_\text{act}$
    - $\Delta G < 0$ exothermic
    - $\Delta G > 0$ endothermic

- Velocity constant of reaction
  - Strongly dependent on T
  - Reaction-specific
  - Arrhenius equation

$$k'_c(T) = k e^{-\boxed{E_\text{act}}/R_\text{g}\boxed{T}}$$

2.4. Transport Phenomena

2.4.1. Diffusion

2.4.2. Viscosity

2.4.3. Transport of Heat

2.4.4. Characteristic Numbers

2.4.2. Viscosity

- Transfer of momentum
  from one plane sliding parallel to another
- Mediated by sandwiched intermediate fluid
- „Internal friction“ of fluid

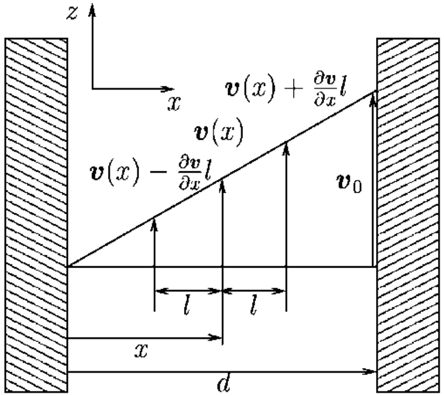


Fig. 2.14. Molecular picture of the viscosity of gases. The left wall is at rest, the right wall located at $z = d$ moves at speed $\boldsymbol{v}_0$ in x-direction and a linear flow profile is assumed in first approximation

- Newton’s law of viscosity

$$\boxed{j_{p,x} = -\eta \frac{\mathrm{d}v_z}{\mathrm{d}x}}$$

- ➢ Relates **flow of axial momentum** $p_z = m v_z$ in **lateral** x-direction from one plane **sliding parallel** to another along z by **viscosity** η
- ➢ Unit of η : $\mathrm{Pa\,s} = \mathrm{kg\,m^{-1}\,s^{-1}}$ or old Poise with $1\ \mathrm{Ps} = 0.1\ \mathrm{Pa\ s}$

2.4.2 Viscosity of Gases

optional

- Assumptions for picture
  - Flow in z-direction
  - Uniform particle density
  - Net flux in x-direction of longitudinal momentum p_z

$$j_{p,x} = \frac{1}{2}\varrho_N m[v_z(x-l_{\mathrm{mfp}}) - v_z(x+l_{\mathrm{mfp}})]\bar{v}_x$$

$$j_{p,x} \simeq -\varrho\frac{\mathrm{d}v_z}{\mathrm{d}x} l_{\mathrm{mfp}} \frac{1}{3} v_T \longleftarrow j_{p,x} = -\eta\frac{\mathrm{d}v_z}{\mathrm{d}x}$$

Newton

- **Viscosity of gases**

$$\boxed{\eta = D\varrho = \frac{1}{3}\varrho_N m v_T l_{\mathrm{mfp}}}$$

$T \uparrow$

- Kinematic viscosity
  - „Momentum diffusivity“

$$\upsilon = \frac{\eta}{\varrho} \overset{\text{id.gas}}{=} \boxed{D}$$

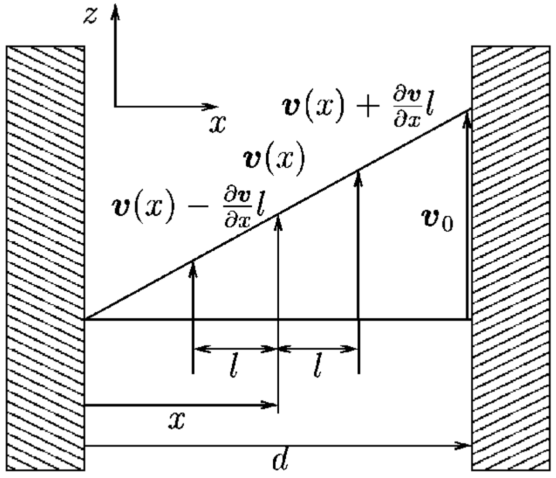
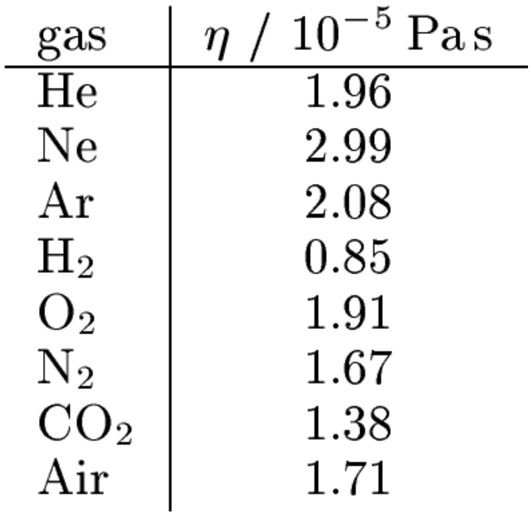


Fig. 2.14. Molecular picture of the viscosity of gases. The left wall is at rest, the right wall located at $z = d$ moves at speed $\boldsymbol{v}_0$ in x-direction and a linear flow profile is assumed in first approximation

2.4.2 Viscosity of Gases

Order: 10^{-5} Pa s



gas	η / 10^{-5} Pa s
He	1.96
Ne	2.99
Ar	2.08
H_2	0.85
O_2	1.91
N_2	1.67
CO_2	1.38
Air	1.71

Table 2.9. Viscosity of gases at $T = 293\,\mathrm{K}$

2.4.2 Viscosity of Liquids

optional

- Liquid
  - “Very dense gas“
  - $l_{\mathrm{mfp}} \approx$ average distance between molecules
  - Macroscopic behavior governed by intermolecular potentials
- Picture
  - Moving A to A‘
  - Activation energy E_0
  - External shear stress σ_{xz}
    - Force *f* on each particle on wall
    - f counteracts motion in negative z-direction
    - Momentum loss transferred to B
    - „Activation energy“ $\Delta E = f\,a/2$

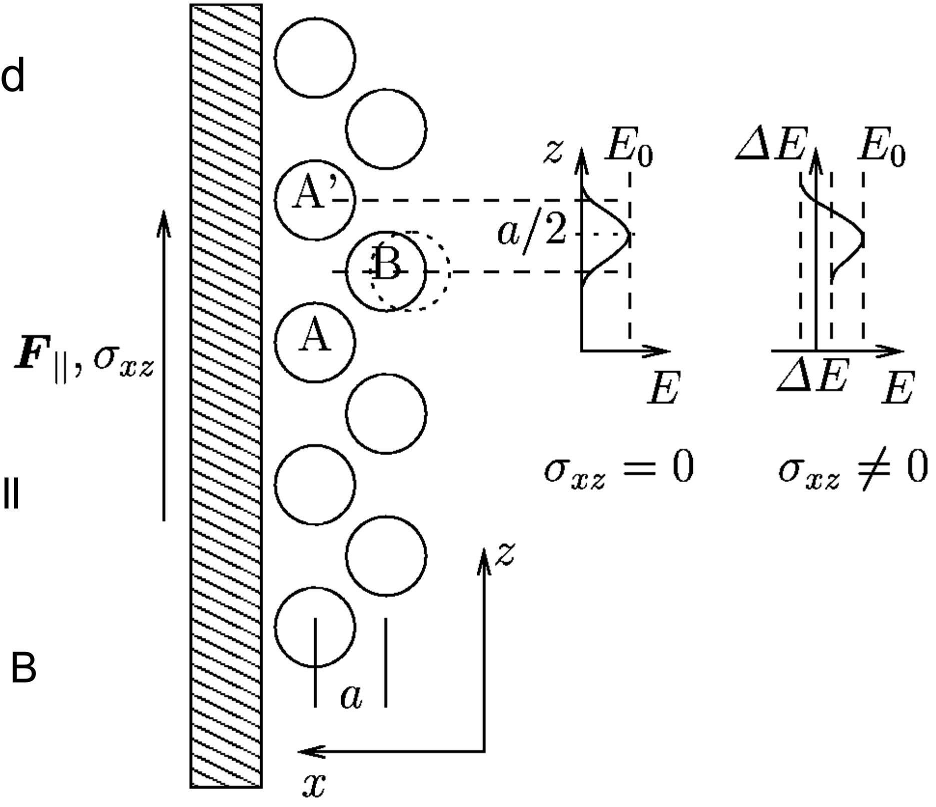


Fig. 2.15. Molecular model for the viscosity of liquids

2.4.2 Viscosity of Liquids

optional

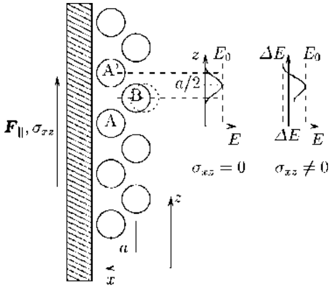


Fig. 2.15. Molecular model for the viscosity of liquids

- Transfer of momentum $\Delta p_z = m\,\Delta v_z$
  - Lateral velocity gradient

$$\frac{\Delta v_z}{\Delta x} = a\nu_{\text{net}}/a = \nu_{\text{net}}$$

  - ν_{net} difference between number of jumps ν_{+z} and ν_{-z} in positive and negative z-direction, respectively
  - Evaluation of gradient (using Boltzmann Ansatz)

$$\begin{aligned}\frac{\Delta v_z}{\Delta x} &= \nu_{+z}^{\text{net}} = \nu_{+z} - \nu_{-z} \\ &= \nu_0 \exp\left(-\frac{E_0 - \Delta E}{k_B T}\right) - \nu_0 \exp\left(-\frac{E_0 + \Delta E}{k_B T}\right) \\ &= \nu_0 \exp\left(-\frac{E_0}{k_B T}\right)\left[\exp\left(\frac{\Delta E}{k_B T}\right) - \exp\left(-\frac{\Delta E}{k_B T}\right)\right] \\ &\simeq \nu_0 \exp\left(-\frac{E_0}{k_B T}\right) 2\frac{\Delta E}{k_B T}\end{aligned}$$

$$\boxed{\frac{\Delta v_z}{\Delta x} = 2\nu_0 \frac{fa}{2k_B T} \exp\left(-\frac{E_0}{k_B T}\right)}$$

ν_0 fundamental oscillation frequency of molecule between neighbors

2.4.2 Viscosity of Liquids

optional

- First order Taylor expansion for $\Delta E / k_B T$

$$\exp\left(\frac{\Delta E}{k_B T}\right) \simeq 1 + \frac{\Delta E}{k_B T}$$

➢ **Viscosity of liquids**

$$\eta = \frac{\sigma_{xz}}{\Delta v_z / \Delta x} = \frac{f / a^2}{2\nu_0 (f a / 2 k_B T) \exp(-E_0 / k_B T)}$$

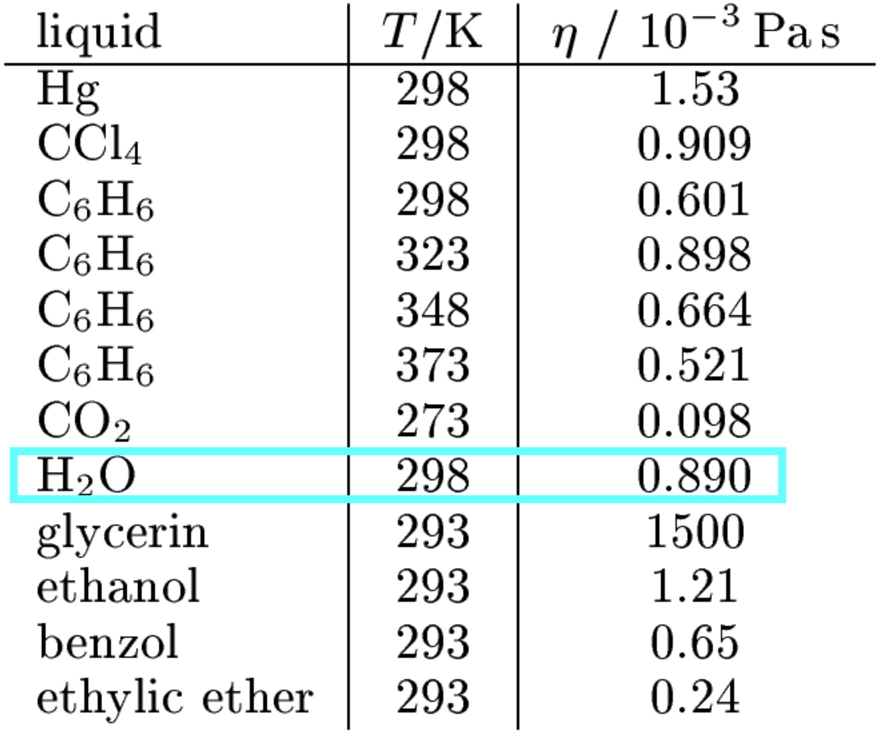
$$\boxed{\eta = \frac{k_B T}{a^3 \nu_0} \exp\left(\frac{E_0}{k_B T}\right)}$$

$T \downarrow$

2.4.2 Viscosity of Liquids

Order: 10^{-3} Pa s

(gases: 10^{-5} Pa s)



liquid	T/K	η / 10^{-3} Pa s
Hg	298	1.53
CCl_4	298	0.909
C_6H_6	298	0.601
C_6H_6	323	0.898
C_6H_6	348	0.664
C_6H_6	373	0.521
CO_2	273	0.098
H_2O	298	0.890
glycerin	293	1500
ethanol	293	1.21
benzol	293	0.65
ethylic ether	293	0.24

Table 2.10. Viscosity of selected liquids at different temperatures

$$\boxed{\eta = \frac{k_B T}{a^3 \nu_0} \exp\left(\frac{E_0}{k_B T}\right)}$$

$T \downarrow$

2.4.2 Newtonian Fluids

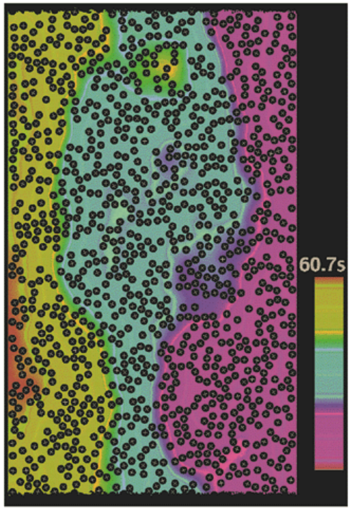
optional

- Newtonian fluids
  - Classical idealized fluids
  - Linear stress-strain relationship
  - I.e., linear relationship between tension tensor σ and tensor of rate of deformation (strain rate) υ

$$\boxed{\sigma \propto \upsilon}$$

    - Viscosity η: constant of proportionality
  - Approximated by gases and „simple“ liquids like water

- Non-Newtonian fluids
  - Dispersions containing large structured macromolecules
  - E.g., polymers and proteins



2.4. Transport Phenomena

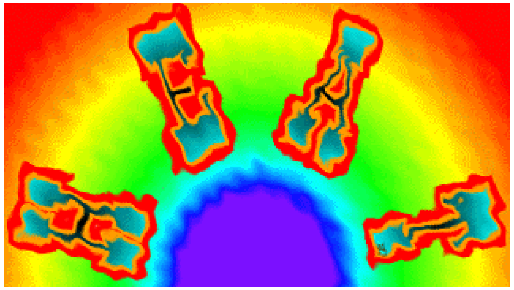
2.4.1. Diffusion

2.4.2. Viscosity

2.4.3. Transport of Heat

2.4.4. Characteristic Numbers

2.4.3. Transport of Heat



- Equilibrium thermodynamics
  - ➢ System seeks uniform temperature distribution
- Inhomogeneous T-distribution
  - ➢ Transport of heat
- Two basic modes
  - ➢ Diffusion
    - Statistical phenomenon (related to entropy)
    - Summarized in macroscopic thermal conductivity λ
  - ➢ Convection
    - Far more complicated
    - Macroscopic ramifications
    - Minor importance in microworld
- Radiation: Another frequently encountered issue
  - ➢ Transfer of thermal energy at interface between two media/phases
  - ➢ No inter-phase diffusion / exchange of particles
  - ➢ Transmission and transition of heat

2.4.3 Thermal Conductivity

- Fourier‘s law
  - Basic equation quantifying diffusive transport of heat

$$\boxed{j_Q = -\lambda \frac{\mathrm{d}T}{\mathrm{d}z}}$$

  - Net flow of energy j_Q
  - Thermal conductivity λ
- Power

$$\boxed{P = j_Q A = \boxed{A}\boxed{\lambda}\boxed{\frac{\mathrm{d}T}{\mathrm{d}z}}}$$

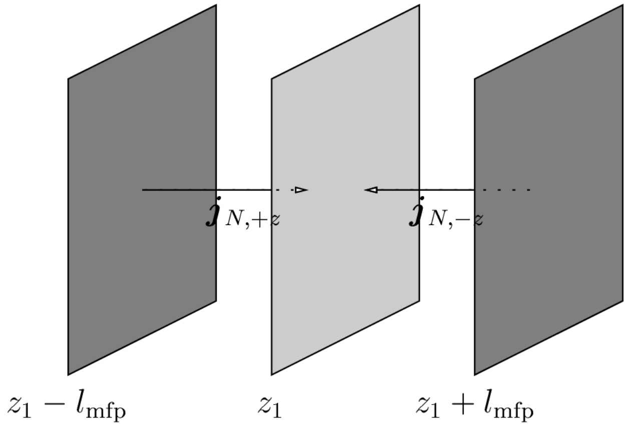
  - Cross-section of flow A
- Relaxation time for temperature gradient
  - Distance d

$$\boxed{\tau = \frac{\boxed{d^2}\varrho\boxed{C_m}}{\boxed{\lambda}}}$$

2.4.3 Thermal Conductivity

optional

- Molecular picture
  - Net energy flow in positive z-direction
  - Using linear terms of Taylor-expansion



$$
\begin{aligned}
j_{Q,+z} &= \frac{1}{6}\varrho_N v_T[\bar{E}_N(z - l_{\mathrm{mfp}}) - \bar{E}_N(z + l_{\mathrm{mfp}})] \\
&= \frac{1}{6}\varrho_N v_T \left\{\left[\bar{E}_N(z) - l_{\mathrm{mfp}}\frac{\partial \bar{E}_N}{\partial z}\right] - \left[\bar{E}_N(z) + l_{\mathrm{mfp}}\frac{\partial \bar{E}_N}{\partial z}\right]\right\} \\
&= -\frac{1}{3}\varrho_N v_T l_{\mathrm{mfp}}\frac{\partial \bar{E}_N}{\partial z}
\end{aligned}
$$

$$
\boxed{j_{Q,+z}} = -\boxed{\left(\frac{1}{3}\varrho_N v_T l_{\mathrm{mfp}}\frac{\partial \bar{E}_N}{\partial T}\right)}\frac{\partial T}{\partial z} = \boxed{-\boxed{\lambda}\frac{\partial T}{\partial z}}
$$

2.4.3 Thermal Conductivity

optional

- Thermal conductivity

$$\boxed{\lambda = D \varrho C_m = \frac{1}{3} \varrho C_m \upsilon_T l_{\mathrm{mfp}}}$$

  - ➢ Approximating on l_{mfp}

$$\lambda = \frac{1}{3\sqrt{2}\sigma_{\mathrm{coll}}} \upsilon_T \frac{\partial \bar{E}_N}{\partial T}$$

  - ➢ Independent of particle density ρ_N for ideal gas

- Heat diffusion coefficient

$$\lambda_D = \frac{\lambda}{\varrho C_m} \overset{\mathrm{id.gas}}{=} \boxed{D} \longleftarrow \quad \upsilon = \frac{\eta}{\varrho} \overset{\mathrm{id.gas}}{=} \boxed{D}$$

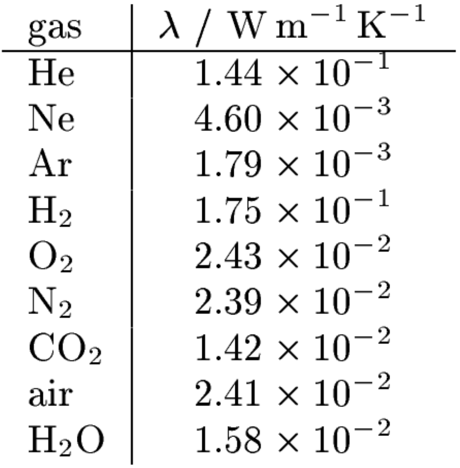
kinematic viscosity

2.4.3 Thermal Conductivity

$$\boxed{\lambda = D\varrho C_m = \frac{1}{3}\varrho C_m v_T l_{\mathrm{mfp}}}$$

- Typical values

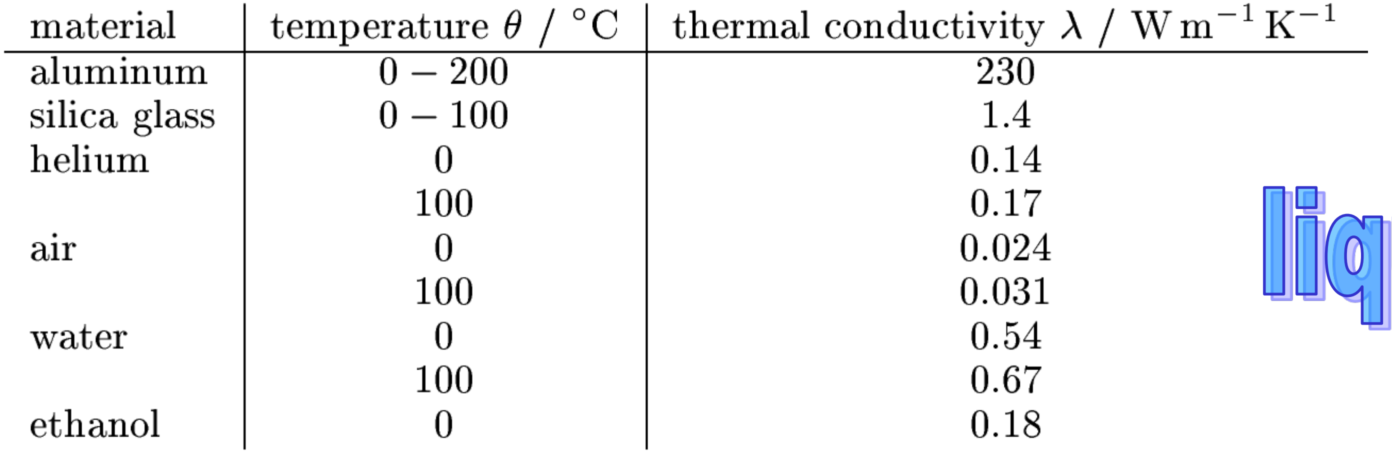
gases



gas	λ / $\mathrm{W\,m^{-1}\,K^{-1}}$
He	1.44×10^{-1}
Ne	4.60×10^{-3}
Ar	1.79×10^{-3}
H_2	1.75×10^{-1}
O_2	2.43×10^{-2}
N_2	2.39×10^{-2}
CO_2	1.42×10^{-2}
air	2.41×10^{-2}
H_2O	1.58×10^{-2}

Table 2.11. Thermal conductivity of gases at $T = 273\,\mathrm{K}$

liquids



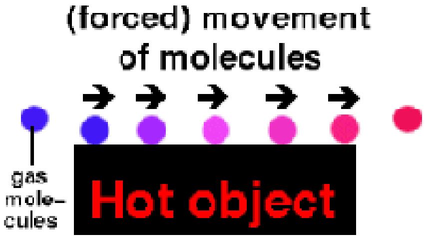
material	temperature θ / °C	thermal conductivity λ / $\mathrm{W\,m^{-1}\,K^{-1}}$
aluminum	0 – 200	230
silica glass	0 – 100	1.4
helium	0	0.14
	100	0.17
air	0	0.024
	100	0.031
water	0	0.54
	100	0.67
ethanol	0	0.18

Table 2.12. Thermal conductivity for solids, liquids and gases

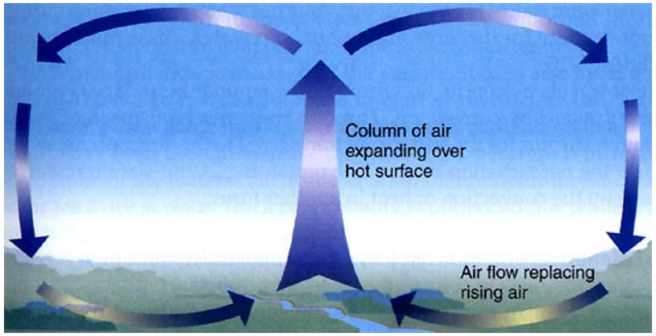
2.4.3 Convection

- Macroscopic transport of particles
  - Characteristic for macro-devices
  - Important for Transport of Heat in liquids and gases

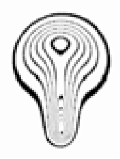
- Forced convection
  - Transport of heat by macroscopic particle flow
  - E.g., by an *external mechanical* source



- Free convection
  - For instance, driven by *buoyancy*

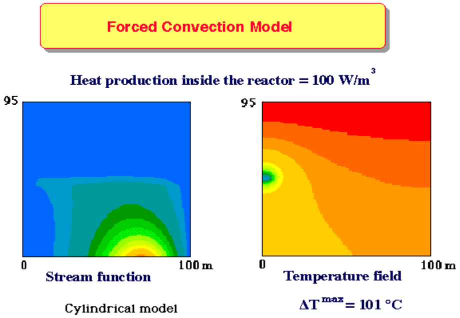


2.4.3 Convection



- Convective currents
  - Stationary or non-stationary flow patterns

- Simulation of convection
  - Often fail to coincide with experimental observations
  - E.g., atmospheric weather and climate



- Free convection of minor importance in microdevices
  - Laminar conditions

2.4. Transport Phenomena

optional

2.4.1. Diffusion

2.4.2. Viscosity

2.4.3. Transport of Heat

2.4.4. Characteristic Numbers

2.4.4. Characteristic Numbers

optional

- Fourier mass number
  - Characterizes diffusion

$$\boxed{Fo_m = \frac{Dt}{l^2}}$$

  - Time *t*, e.g., residence in chemical reaction chamber
  - Typical diffusive time scale $t_D = l^2/D$

- Schmidt number
  - Relates viscosity and diffusion

$$\boxed{Sc = \frac{\eta}{\varrho D} = \frac{\nu}{D} \overset{\text{id. gas}}{=} 1}$$

  - Roughly 0.8 for gases

2.4.4. Characteristic Numbers

optional

- Fourier number
  - Diffusion of heat
  - Stored thermal energy

$$Fo = \frac{\lambda t}{\varrho C_m l^2} = \frac{\lambda_D t}{l^2}$$

- Prandtl number
  - Ratio between momentum diffusivity (via dynamic viscosity η) and heat diffusivity (via thermal conductivity λ)

$$Pr = \frac{C_m \eta}{\lambda} = \frac{\upsilon}{\lambda_D} \overset{\text{id.gas}}{=} 1$$

  - Specific heat capacity C_m
  - Typically 3 to 300 for liquids and 0.7 to1.0 for gases

Summary: Important Formulas I

Diffusion constant

$$D = \frac{1}{3} v_T l_{\mathrm{mfp}}$$

Boltzmann distribution: Energy E

$$\frac{\varrho_{N,1}}{\varrho_{N,0}} = \mathrm{e}^{-\frac{\Delta E}{k_\mathrm{B} T}}$$

$$\text{tail fraction} = \frac{\int_{E_0}^{\infty} f(E)\mathrm{d}E}{\int_0^{\infty} f(E)\mathrm{d}E} = \frac{2}{\sqrt{\pi}} \sqrt{\frac{E_0}{k_\mathrm{B} T}} \mathrm{e}^{-E_0/k_\mathrm{B} T}$$

Maxwell distribution: particle velocity v

$$f(v)\mathrm{d}v = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_\mathrm{B} T} \right)^{3/2} v^2 \mathrm{e}^{-mv^2/2k_\mathrm{B} T} \mathrm{d}v$$

Summary: Important Formulas II

Diffusion scaling $l_D = \sqrt{2Dt}$ $t_D = \frac{l^2}{2D}$

Viscosity (for ideal gas) $\eta = D\varrho$

Kinematic viscosity $\upsilon = \frac{\eta}{\varrho} \overset{\text{id.gas}}{=} D$

Thermal conductivity $j_Q = -\lambda \frac{\mathrm{d}T}{\mathrm{d}z}$

Thermal relaxation time $\tau = \frac{d^2 \varrho C_m}{\lambda}$