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3. Physics of Microfluidic Systems

1. Navier-Stokes Equations
2. **Laminar and Turbulent Flow**
3. Fluid Dynamics
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6. Interfacial Surface Tension
7. Electrokinetics

3.2. Laminar and Turbulent Flow

1. **Critical Reynolds Number**
2. Shear-Driven Laminar Flow
3. Taylor-Couette Flow
4. Laminar PDF through a Tube
5. Laminar PDF through a Gap
6. Irrotational Flow
7. Centrifugal-Force Driven Flow
8. Effects in Laminar Flows
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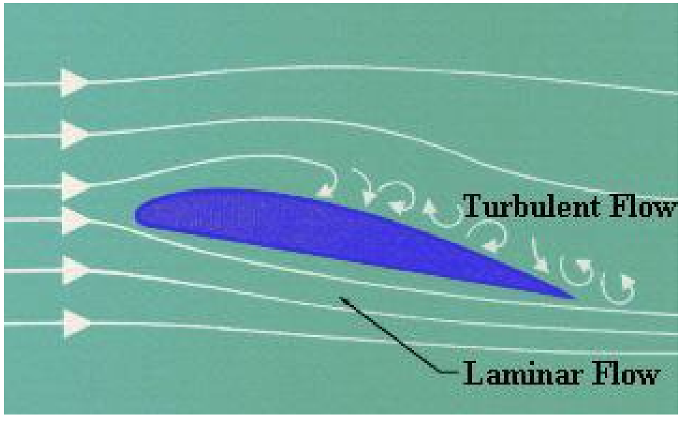
3.2.1. Critical Reynolds Number

<u>Three types of flow regimes:</u>



- Laminar
  - Low flow velocities
  - Smooth sliding of adjacent layers
  - Field of velocity vectors constant in time
- Turbulent
  - Curling of field lines
  - Mixing between adjacent layers
  - „Unpredictable" development of field of velocity vectors
  - Flow patterns increasingly turbulent towards high velocities
  - Sometimes laminar flow preserved up to higher velocities
- Periodic flow
  - 3rd flow regime
  - Surface waves
  - Acoustic waves
- All three flow types solutions of NS-equation

3.2.1. Perturbation Analysis

optional

- Transition from laminar to turbulent flow regime
  - Mathematical perturbation analysis
  - Prediction whether velocity distribution belongs to distinct flow regime
- Ansatz
  - Known solution of NS-equation (guessed or measured)
  - Superimposing small perturbation

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_0(\boldsymbol{r},t) + \delta\boldsymbol{v}$$

  - Product of
    - Amplitude
    - Oscillatory factor
    - Exponential term

$$\delta\boldsymbol{v} = \boldsymbol{A}(\boldsymbol{r})\mathrm{e}^{(\mathrm{i}\omega+\gamma)t} \text{ with } |\delta\boldsymbol{v}| \ll |\boldsymbol{v}|$$

3.2.1. Perturbation Analysis

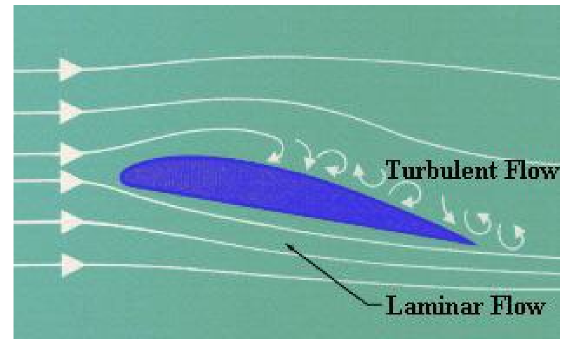
optional

$$\delta \boldsymbol{v} = \boldsymbol{A}(\boldsymbol{r})\mathrm{e}^{(\mathrm{i}\omega+\gamma)t} \text{ with } |\delta \boldsymbol{v}| \ll |\boldsymbol{v}|$$

- Properties
  - Locally varying amplitude A
  - ω and γ constant for given problem
- Insertion of perturbed solution in NS as initial velocity field
  - Result: First order equations of ω and γ
  - Sign of γ indicates decay of perturbation into $\boldsymbol{v}_0$

3.2.1. Critical Reynolds Number

- Condition $\gamma = 0$ defines critical Reynolds number Re^*
- $Re < Re^*$
  - Perturbations damped in time
- $Re > Re^*$
  - Exponential growth of perturbations in time
  - Perturbation theory not valid
  - „Unpredictable“ behavior of velocity field
- Transition point $Re = Re^*$
  - Flow oscillates between two flow regimes
- As Re increases further, turbulent character of flow increases



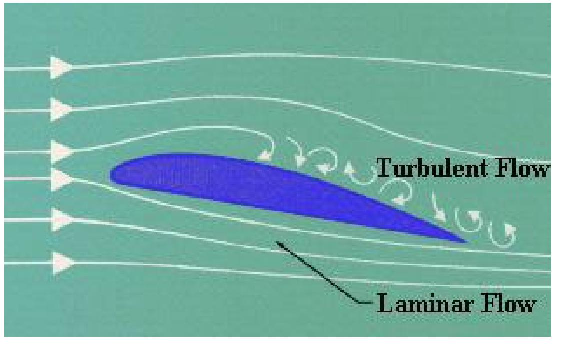
3.2.1. Critical Reynolds Number

- Re^* ranges between 1 and 100,000
- Re^* depends on
  - Material properties (density ϱ, viscosity η)
  - Boundary conditions $\partial\Omega$
  - Critical velocity

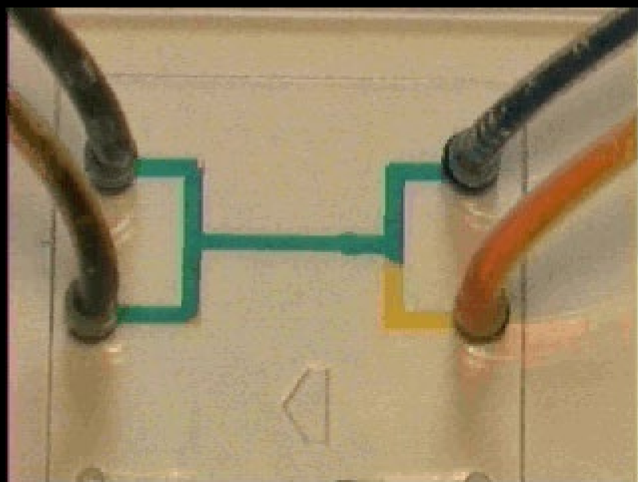
$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v} \tilde{l}}{\upsilon}$$

$$v^* = Re^* \upsilon / l$$

- Specific Re^* geometries
  - Sphere: 2320
  - Flow parallel to plate: $Re^* = 500{,}000$
- Microdevice
  - $Re \approx 1 \ll Re^*$



3.2. Laminar Flow



3.2.1. Critical Reynolds Number

optional

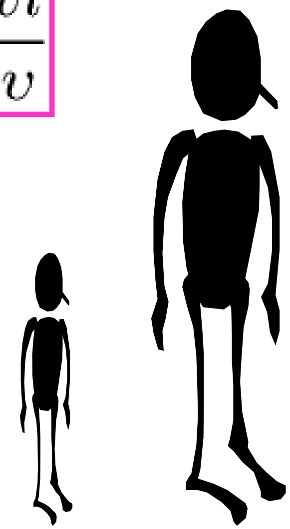
$$Re = \frac{\varrho_\infty \tilde{v}\tilde{l}}{\eta} = \frac{\tilde{v}\tilde{l}}{\upsilon}$$

- Transition point also depends on
  - Initial velocity field
  - Experimental environment
- Domain $Re < Re^*$
  - No survival of initial turbulences
- Domain $Re > Re^*$
  - Laminar flow still possible under certain conditions
  - Turbulences hampered by
    - Smooth walls
    - Smooth endings at orifices
  - Laminar conditions up to $Re = 100{,}000$
  - $Re > 100{,}000$
    - Thermal motion of molecules sufficient to trigger transition to turbulence

3.2.1. Shift of Re* in MF Systems

$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v} \tilde{l}}{\upsilon}$$

- In MF-systems
  - Channel diameter 100 µm
  - Flow velocity $v = 10\ \mathrm{mm\ s^{-1}}$
  - Flow rate $Q = A\ v = 6\ \mu\mathrm{l\ min^{-1}}$
  - $Re \sim 1 << Re^* \sim 2300$



- Always laminar flow in MF-systems?
  - Departure of Reynolds theory in microworld
  - Much higher surface-to-volume ratios
  - Higher sensitivity to surface roughness
  - Reliable results only with smooth surfaces and stabilized pumping
  - Some results indicate early departure from laminar flow regime
    - $300 < Re^* < 900$ or $200 < Re^* < 700$
  - Avoiding artificially induced nucleation of turbulence
    - Smooth walls
    - Steady pumping

3.2. Laminar and Turbulent Flow

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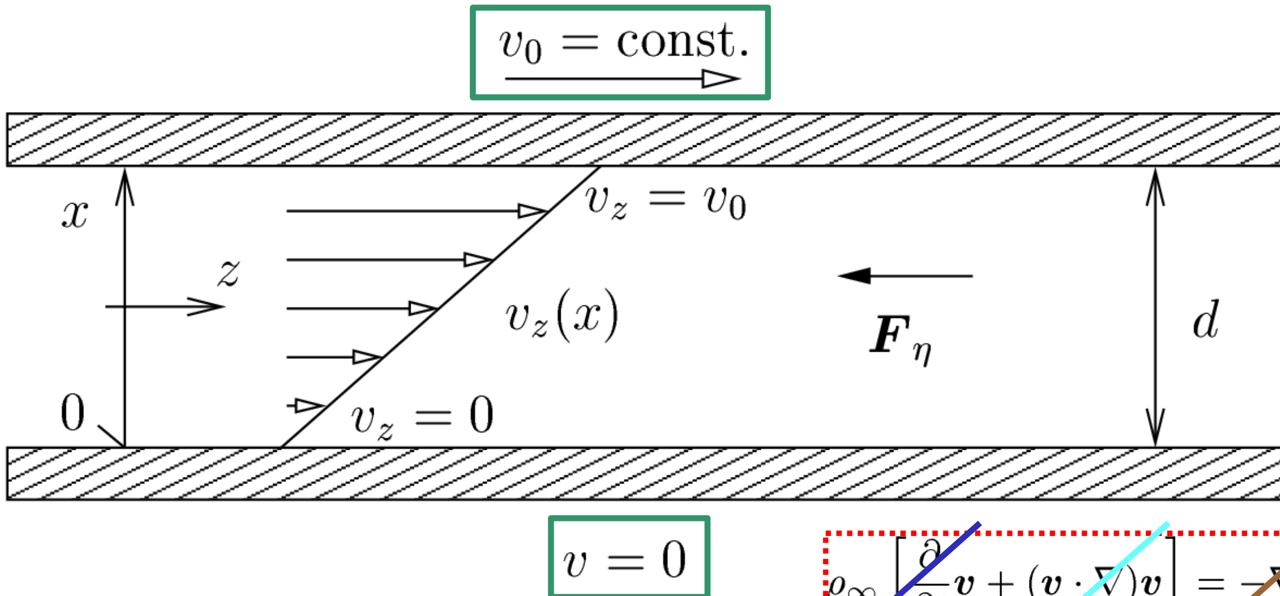
3.2.2. Shear-Driven Laminar (Couette) Flow

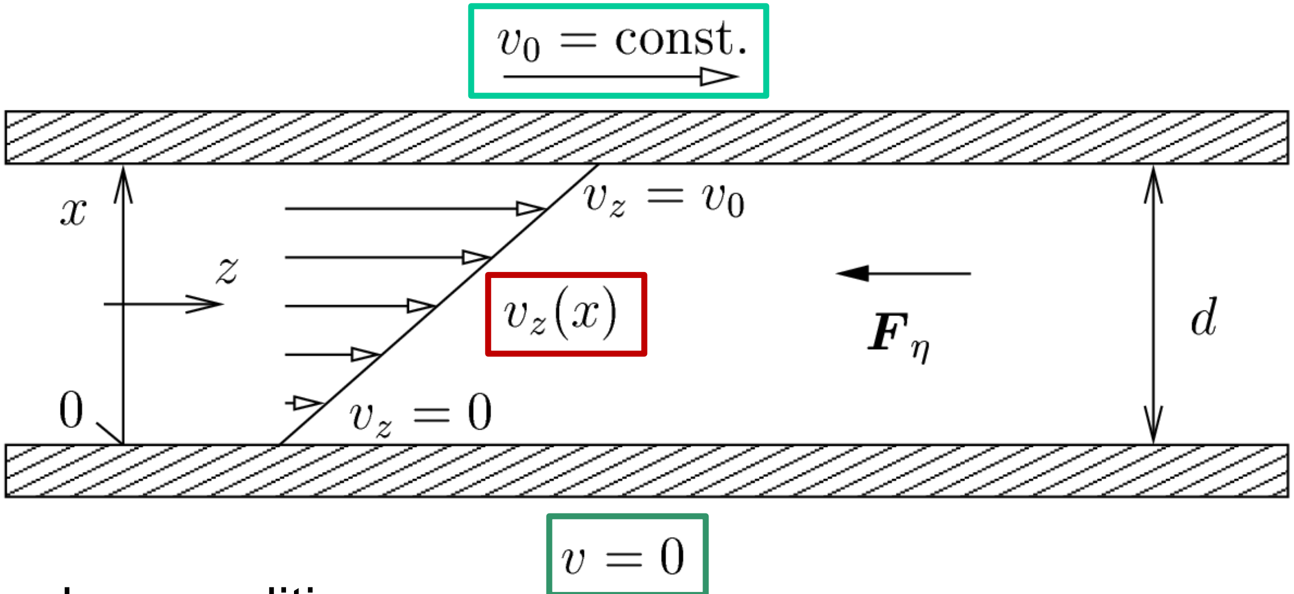


$$\varrho_\infty \left[\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \eta\nabla^2\boldsymbol{v} + \varrho_\infty \boldsymbol{g}$$

- Flow imposed on fluid by mere shear forces
  - Vanishing gravity $g = 0$
  - No external pressure gradient $\nabla p = 0$
- Viscous fluid sandwiched between two plates
  - Wall at $x = 0$ at rest
  - Wall at $x = d$ moving at speed $v_0 = \text{const.}$ in z-direction
  - Stationary flow
  - Constant velocity on axial direction
  - Viscous force density

$$\eta \frac{\partial^2 v}{\partial x^2} = 0$$

3.2.2. Shear-Driven Laminar (Couette) Flow



- Boundary conditions

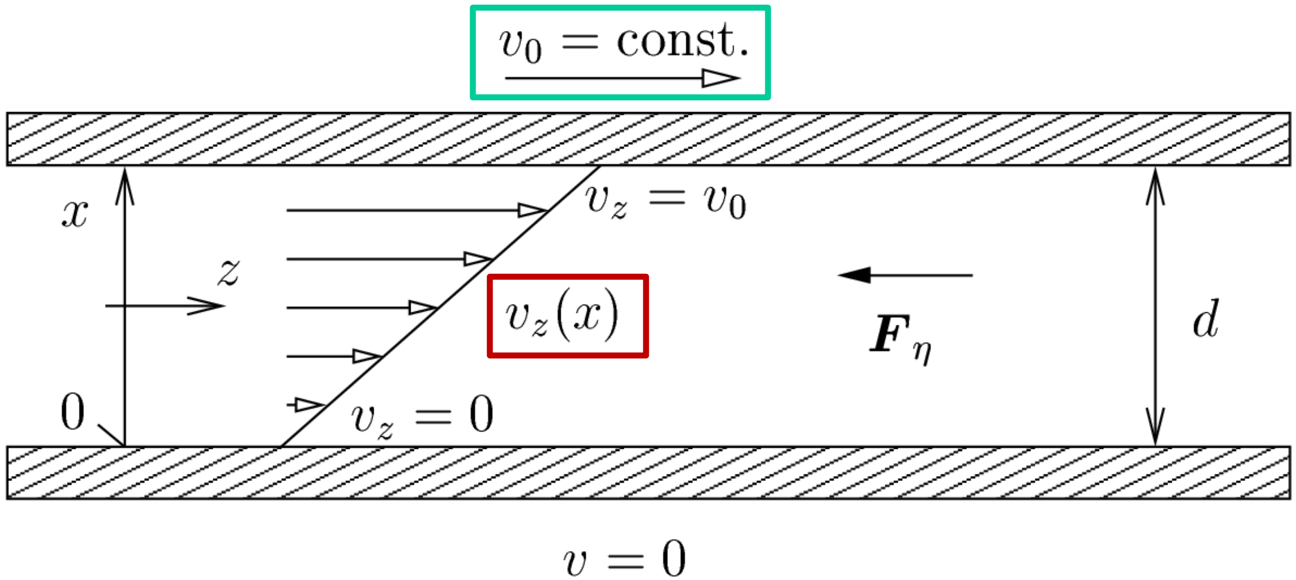
$$v_z(x = 0) = 0$$

$$v_z(x = d) = v_0$$

- Linear flow profile

$$v_z(x) = \frac{x}{d} v_0$$

3.2.2. Shear-Driven Laminar (Couette) Flow



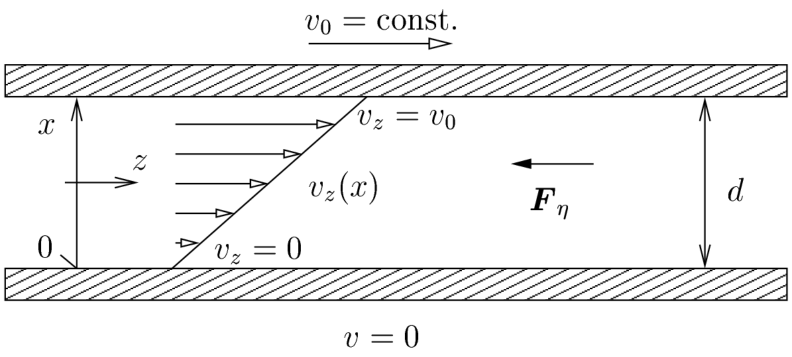
- Viscous force
  - Flow density $j_{p,x}$ of axial momentum $p_z = m v_z$ in lateral x-direction
  - Total force

$$j_{p,x} = \eta \cdot \frac{dv}{dx} = \eta \cdot \frac{v_0}{d}$$

$$F_\eta = j_{p,x} \cdot A = \eta \cdot \frac{v_0}{d} \cdot A$$

3.2.2. Prandtl Boundary Layer

optional



- d increases until $Re > Re^*$
  - Turbulent regime
- Laminar regime restricted to
  - Tiny layer $\delta < d$ near moving body
- "Prandtl layer"
  - Diffusion-limited mass and heat transfer
  - Decisive impact on mass and heat exchange in macrosystems
- Estimate for thickness δ by energy analysis
  - Viscous work

$$|\boldsymbol{f}_\eta| \simeq \eta \frac{2v_{\max}}{d^2}$$

$$W_\eta = F_\eta l = \frac{\eta A v_0 l}{\delta}$$

  - Spent when body traveling at v_0 covers distance of its own length l

3.2.2. Prandtl Boundary Layer

optional

- Setting adjacent fluid into motion requires kinetic energy

$$E_{\mathrm{kin}} = \frac{1}{2}\varrho A \int_0^\delta \left(\frac{v_0 z}{\delta}\right)^2 \mathrm{d}z = \frac{1}{6} A \varrho v_0^2 \delta$$

  - Assuming linear flow profile within Prandtl layer
  - Setting equal kinetic energy and viscous work yields

$$\delta = \sqrt{\frac{6\eta l}{\varrho v_0}} = \sqrt{\frac{6 l d}{Re}}$$

  - Typical MF-values: $l = 1$ cm, $d = 100$ µm and $Re = 1$
  - $\delta \approx 1$ cm >> d
  - **Fully developed Prandtl layer therefore not found in MF systems**
  - Attention
    - *Re* increases with speed of flow

3.2. Laminar and Turbulent Flow

optional

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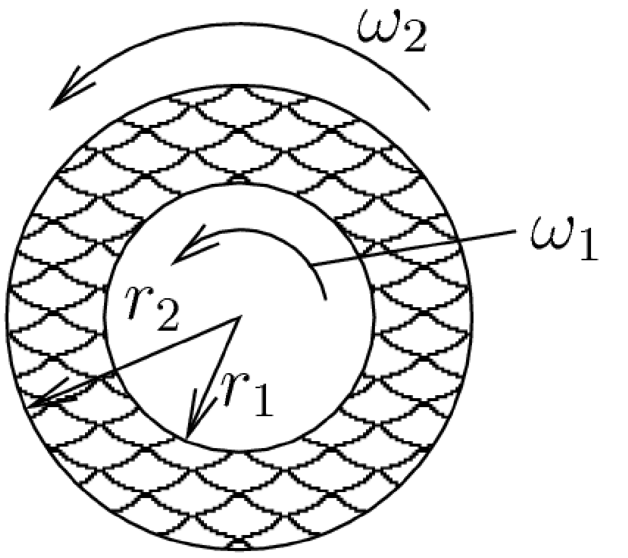
3.2.3. Couette Flow

optional

- Azimuthal symmetry
  - Purely azimuthal fluid motion
  - Cylindrical coordinates (r, ϕ, z)
  - Velocity field $v(r)$
  - Pressure distribution p
  - Symmetry reduces NS-equations and continuity equation to



$$\varrho \frac{v_\phi^2}{r} = \frac{\partial p}{\partial r}$$

$$v_\phi\big|_{r_1} = \omega_1 r_1$$
$$v_\phi\big|_{r_2} = \omega_2 r_2$$

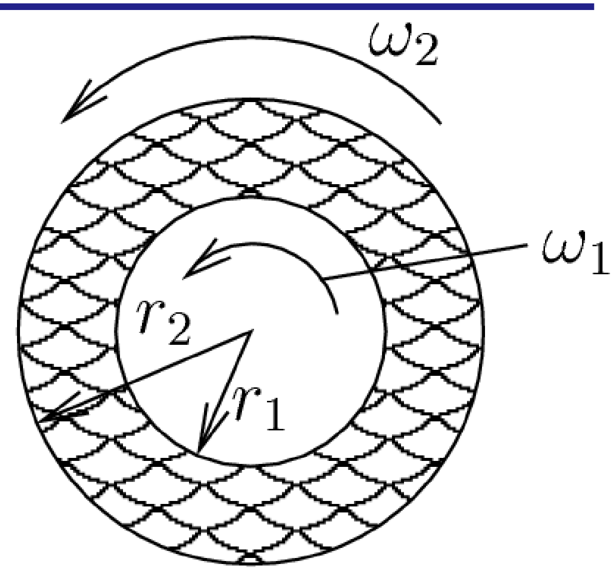
$$0 = \eta \left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} \right)$$

3.2.3. Couette Flow

optional

- Azimuthal symmetry
  - Purely azimuthal fluid motion
  - Cylindrical coordinates (r, ϕ, z)
  - Velocity field $v(r)$
- Ansatz

$$v_\phi = A_i r^n$$



- Solution

small

$$v_\phi(r) = A_1 r + A_2 r^{-1}$$

$\sim r$

$$p(r) = \varrho\left(\frac{1}{2}A_1^2 r^2 + 2A_1 A_2 \ln r - \frac{1}{2}A_2^2 r^{-2}\right)$$

3.2. Laminar and Turbulent Flow

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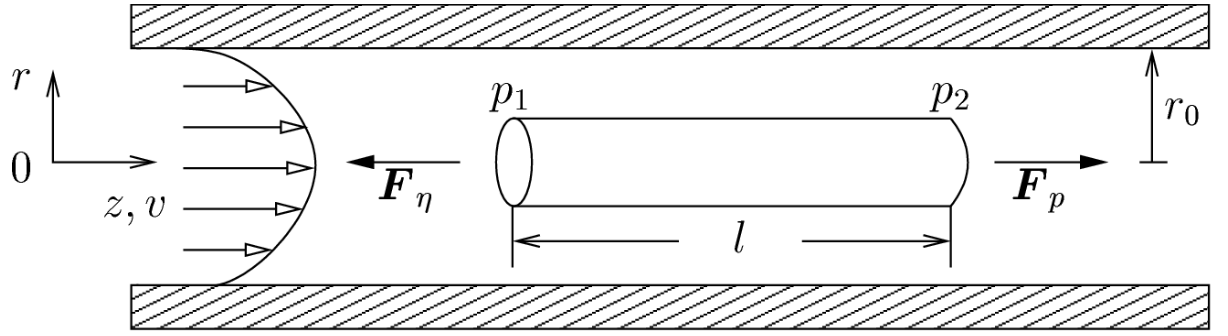
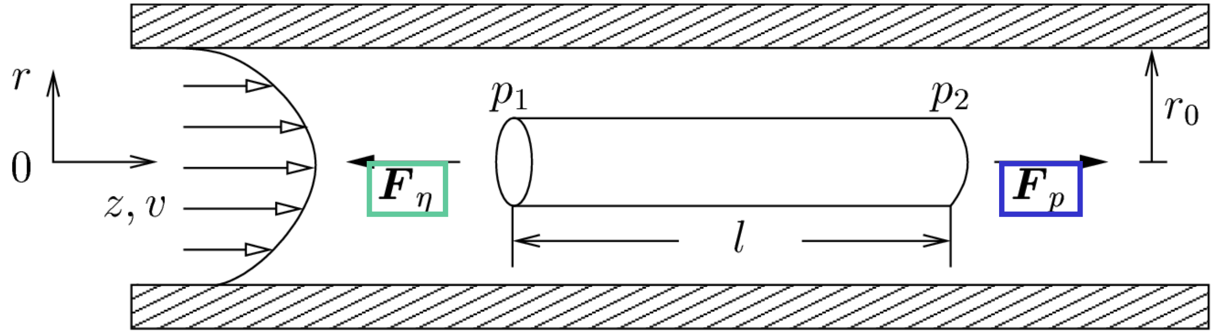
3.2.4. Laminar PDF through Tube



$$\varrho_\infty \left[\frac{\partial}{\partial t} \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right] = -\nabla p + \eta \nabla^2 \boldsymbol{v} + \varrho_\infty \boldsymbol{g}$$

- Pressure-driven flow
  - Important phenomenon in nature
  - E.g., transport of nutrients in plants and animals by heart
- Law of Hagen-Poiseuille (~1840)
  - Pressure drop
  - Throughput
- Symmetry
  - Parabolic flow profile
  - Cylindrical symmetry

3.2.4. Laminar PDF through Tube



- Pressure forces

$$\boldsymbol{F}_p = (p_1 - p_2)\pi r^2 = \pi r^2 \Delta p$$

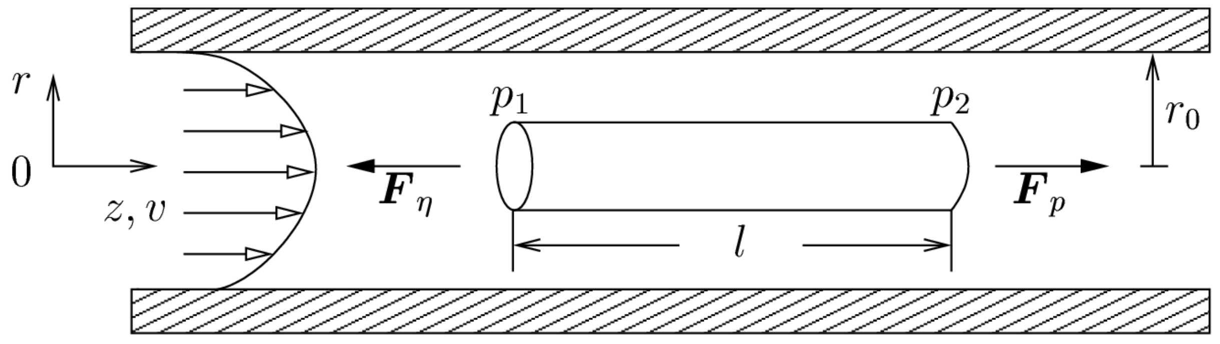
- Viscous forces

$$\boldsymbol{F}_\eta = \eta A_r \frac{\mathrm{d}v_z}{\mathrm{d}r}$$

- Relationship for stationary flow $\frac{dv_z}{dt} = 0$

$$F_p = F_\eta \longrightarrow \pi r^2 \Delta p = -2\pi r l \eta \frac{\mathrm{d}v_z}{\mathrm{d}r}$$

3.2.4. Laminar PDF through Tube



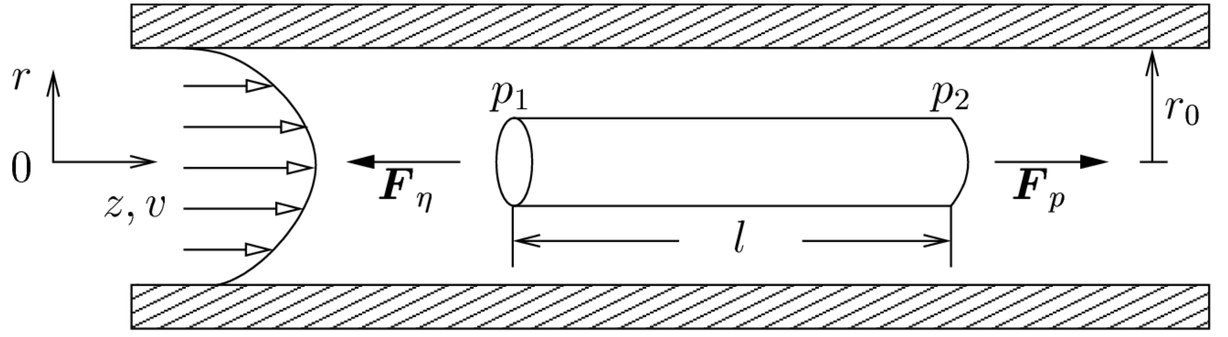
- Integration
  - Extension of auxiliary cylinder of radius r to tube radius r_0

$$v_z(r) - v_z(r_0) = \frac{\Delta p}{2\eta l} \int_r^{r_0} r' \mathrm{d}r' = \frac{\Delta p}{4\eta l}(r_0^2 - r^2)$$

- Flow velocity profile

$$v_z(r) = \frac{\Delta p}{4\eta l}(r_0^2 - r^2)$$

3.2.4. Laminar PDF through Tube



- Maximum velocity (in center at $r = 0$)

$$v_{\mathrm{max}} = \frac{\Delta p}{4\eta l} r_0^2$$

$$v_z(r) = \frac{\Delta p}{4\eta l}(r_0^2 - r^2)$$

- Minimum velocity (at wall at $r = r_0$)

$$v(r = r_0) = 0$$

3.2.4. Flow Rate

- Volumetric flow I_V determined by integration of $v_z(r)\,\mathrm{d}A$ over r_0

$$I_V = \frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^{r_0} v_z(r) 2\pi r \mathrm{d}r = \frac{\pi \Delta p}{2\eta l}\left(\int_0^{r_0} r_0^2 r \mathrm{d}r - \int_0^{r_0} r^3 \mathrm{d}r\right)$$

$$I_V = \frac{\pi}{8\eta}\frac{\Delta p}{l} r_0^4$$

- **Law of Hagen-Poiseuille**
  - I_V scales with r^4
  - Pressure gradient $\Delta p / l$
  - Viscosity η
- Average velocity

$$\bar{v}_z = \frac{\Delta p}{8\eta l} r_0^2$$

$$v = \frac{\eta}{\varrho} \overset{\text{id.gas}}{=} D$$

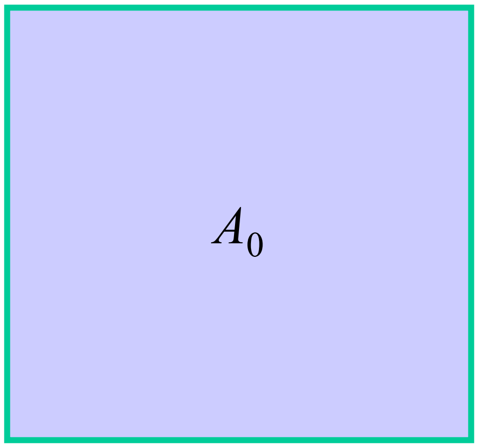
$$Re = \frac{2 I_V}{\pi r_0 v}$$

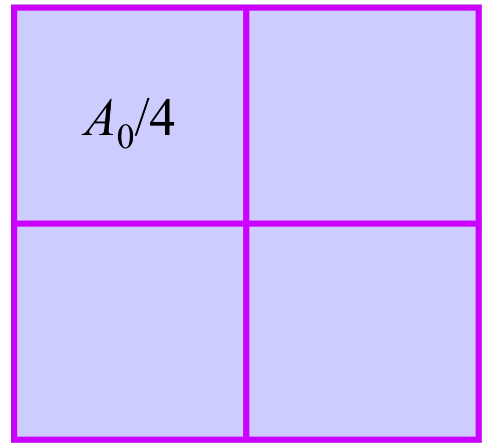
- Alternative expression for Reynolds number

3.2.4. Throughput

$$I_V \sim r^4 = A^2$$

Hagen-Poiseuille





$$I_{0,V} \sim A_0^2$$

$$I_V \sim 4\,(A_0\,/\,4)^2 = ¼\, I_{0V}$$

$$I_V \sim N\,(A_0\,/\,N)^2 = (1/N)\;\; I_{0V}$$

3.2.4. Hydraulic Diameter

optional

- Based on law of Hagen-Poiseuille for cylindrical geometry
- PDF through duct with non-circular cross-section
- Equivalent hydraulic diameter

$$d_{\mathrm{hd}} = \frac{4 \cdot (\mathrm{cross-sectional\ area})}{\mathrm{wetted\ perimeter}}$$

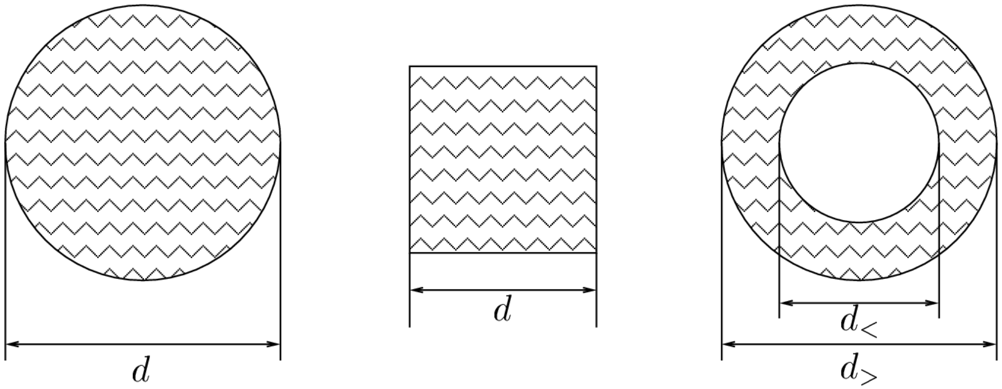


Fig. 3.9. Schematic for values entering hydraulic diameter d_{hd} of circular tube, a square tube and two concentric tubes

3.2.4. Hydraulic Diameter

optional

- Round tube

$$d_{\mathrm{hd}} = \frac{4 \cdot \frac{\pi d^2}{4}}{\pi \cdot d} = d$$

- Square tube
  - Edge length

$$d_{\mathrm{hd}} = \frac{4d^2}{4d} = d$$

- Annular geometry

$$d_{\mathrm{hd}} = \frac{4 \cdot \left\{ \frac{\pi d_>^2}{4} - \frac{\pi d_<^2}{4} \right\}}{\pi d_> + \pi d_<} = d_> - d_<$$

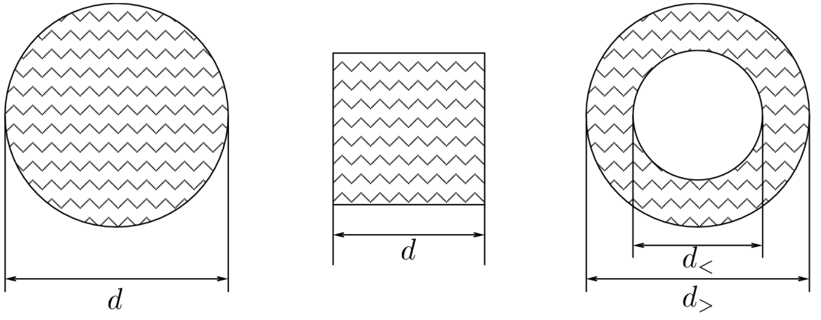
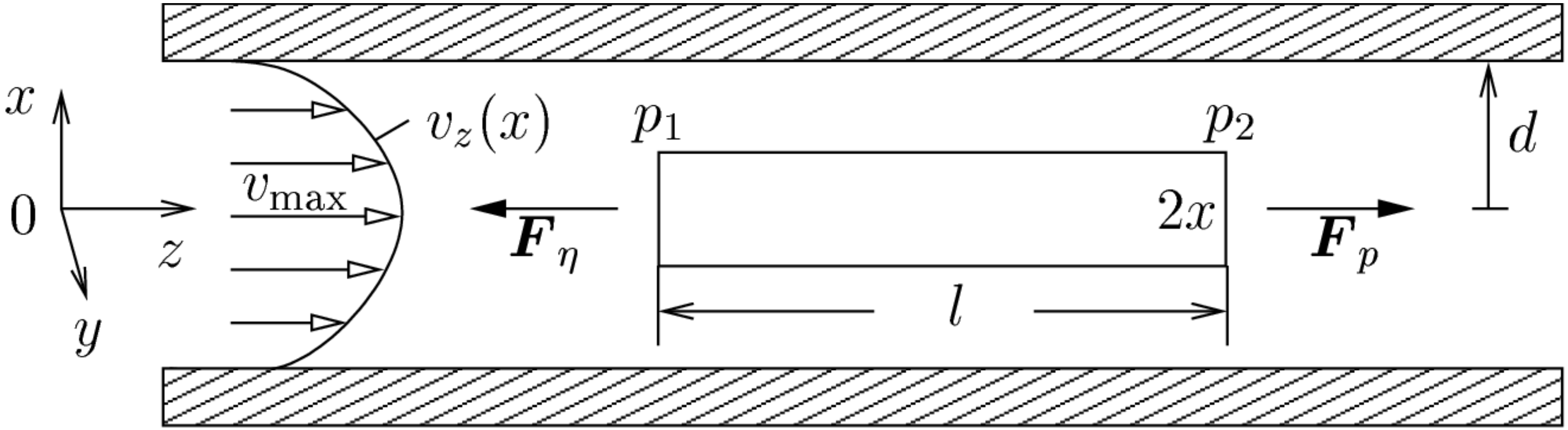


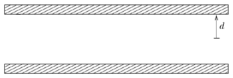
Fig. 3.9. Schematic for values entering hydraulic diameter d_{hd} of circular tube, a square tube and two concentric tubes

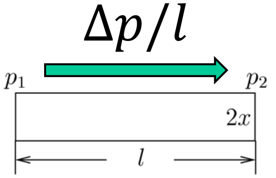
3.2. Laminar and Turbulent Flow

3.2.5. Laminar PDF through Gap



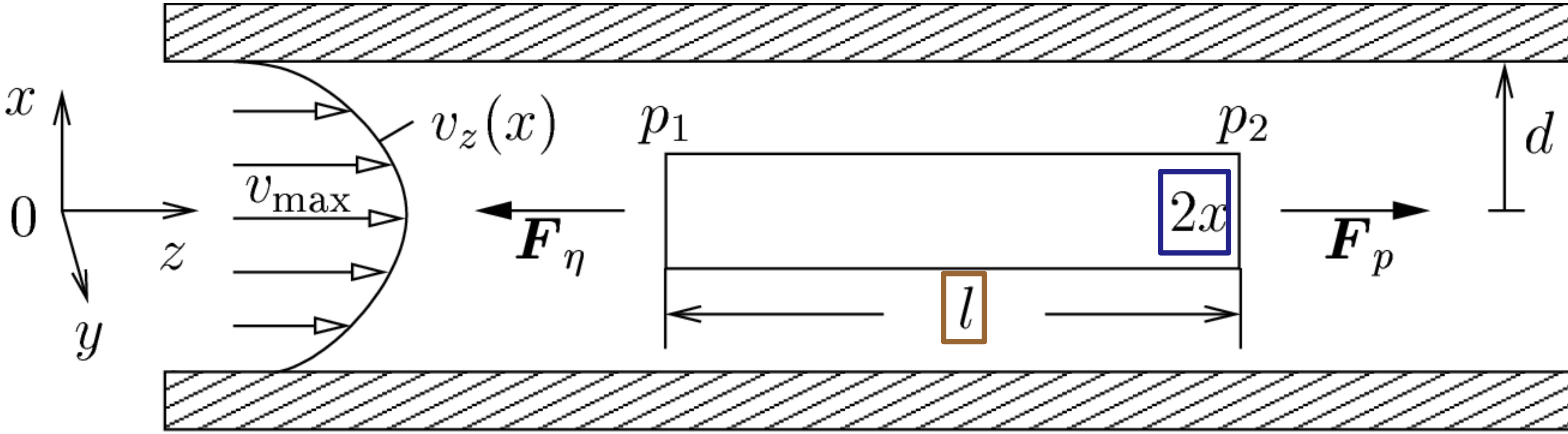
- Pressure-driven flow
  - No (external) shear or volume forces
  - Parallel plates
  - Laminar regime
  - Pressure gradient antiparallel to direction of flow
- No-slip conditions





$$v_z(x = d) = v_z(x = -d) = 0$$

3.2.5. Laminar PDF through Gap



- Rectangular element
  - Width $2x$
  - Length l
  - Depth b (into plane of drawing)
  - Cross section $A_x = b\, l$
  - Fore-part $A_z = 2\, x\, b$
- Total velocity gradient across element $2\, dv / dx\, |_{+/-x}$

3.2.5. Laminar PDF through Gap

- Differential relationship

$$\left.\frac{\mathrm{d}v}{\mathrm{d}x}\right|_{\pm x} = \frac{1}{\eta}\frac{\mathrm{d}p}{\mathrm{d}z}x$$

- Parabolic flow profile

$$v(x) = v_{\max} - \frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}z}x^2 = v_{\max} - \frac{\Delta p}{2\eta l}\boxed{x^2}$$

- Peak velocity

$$\boxed{v_{\max} = \frac{\Delta p}{2\eta l}d^2}$$

- Overall volume flow rate I_V per channel width y

$$\boxed{\frac{I_V}{y}} = 2\int_0^d v(x)\mathrm{d}x = \frac{\boxed{h^3}}{12\eta}\frac{\mathrm{d}p}{\mathrm{d}z}$$

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3.2.6. Irrotational Flows

optional

- Vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$$

  - Vanishes for irrotational flows

- Vector identity

$$\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \boldsymbol{v}) = 0$$

  - Vanishing divergence of vorticity

- For vanishing vorticity, i.e. irrotational flow, $\boldsymbol{v}$ can be written

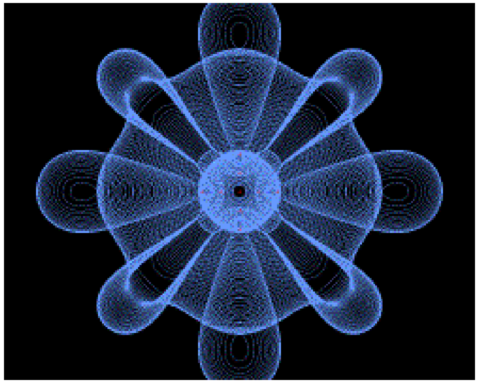
$$\boldsymbol{v} = \nabla \Phi$$

"gradient of a scalar potential Φ"

3.2.6. Potential Flow Theory

- Basic building blocks
  - Set of special flow schemes
  - Analogous to multipole concept in electrodynamics

- Mathematical point of view
  - Special instances of Green's function



3.2.6. Velocity Potentials (2-dim.)

optional

- Simplification
  - 2-dim. velocity field $\boldsymbol{v} = (v_x, v_y)$
- Velocity potential ϕ
  - Scalar
- Stream function ψ
  - Scalar

$$v_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}$$
$$v_y = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

Flow	velocity potential Φ	stream function Ψ	
uniform stream	$xv_x + yv_y$	$yv_x - xv_y$	
source or sink	$\frac{\hat{I}_m}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}$	$\frac{\hat{I}_m}{2\pi} \tan^{-1} \frac{y-y_0}{x-x_0}$	monopole
doublet	$\frac{B_x(x-x_0)+B_y(y-y_0)}{\lvert \boldsymbol{r}-\boldsymbol{r}_0 \rvert^2}$	$\frac{B_y(x-x_0)+B_x(y-y_0)}{\lvert \boldsymbol{r}-\boldsymbol{r}_0 \rvert^2}$	dipole
line vortex	$\frac{\Gamma}{2\pi} \tan^{-1} \left(\frac{y-y_0}{x-x_0} \right)$	$-\frac{\Gamma}{2\pi} \ln \lvert \boldsymbol{r} - \boldsymbol{r}_0 \rvert$	

Table 3.1. Velocity potentials Φ and stream functions Ψ for 2-dimensional irrotational flows

3.2.6. Hele-Shaw Table

optional

- Visualization of basic 2-dim. flows
- Uniform stream over floor to drain
- Bottles
  - Raised or lowered to adjust gravitational pressure
  - Connected to through holes
- 2-dim. flow (top view)
- Sources and drains (monopoles)
  - Holes
- Doublets (dipoles)
  - Source and sink very close to each other
  - Bottles spaced by same distance above and below floor
- Sometimes transparent cover to ensure uniform depth

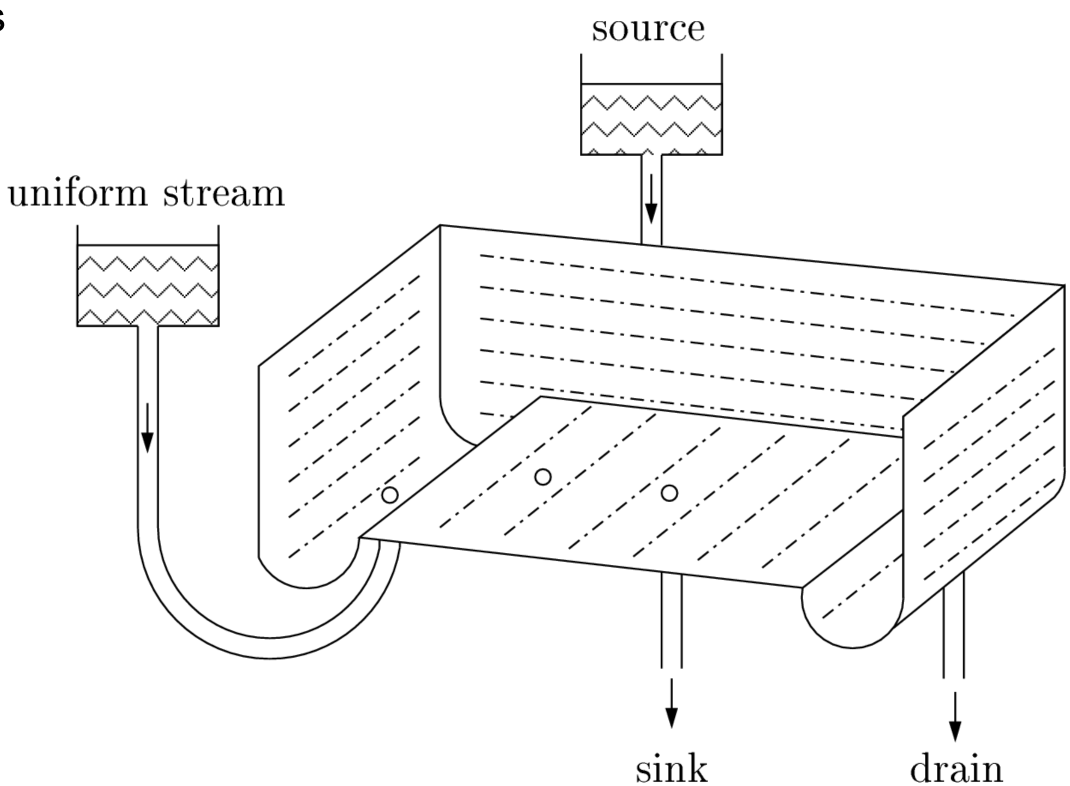


Fig. 3.12. Hele-Shaw table visualizing basic 2-dimensional flow schemes

3.2.6. Bernoulli Equation

- Continuity equation
- Frictionless, irrotational flow

$$0 = \nabla \cdot \boldsymbol{v} = \nabla^2 \Phi$$

- Navier-Stokes
  - ➢ Rewritten

$$\varrho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla |\boldsymbol{v}|^2 \right) = -\nabla p + \varrho \boldsymbol{g}$$

  - ➢ Using vector analysis

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = \left(\frac{1}{2} \nabla |\boldsymbol{v}|^2 \right) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})$$

vanishing vorticity

- General form of Bernoulli

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\boldsymbol{v}|^2 + |\boldsymbol{g}| h + \frac{p}{\varrho} = f(t)$$

- Bernoulli
  - ➢ Stationary conditions
  - ➢ Neglect gravity g
  - ➢ Integration in space

$$p + \frac{\varrho}{2} v^2 = \text{const.}$$

3.2. Laminar and Turbulent Flow

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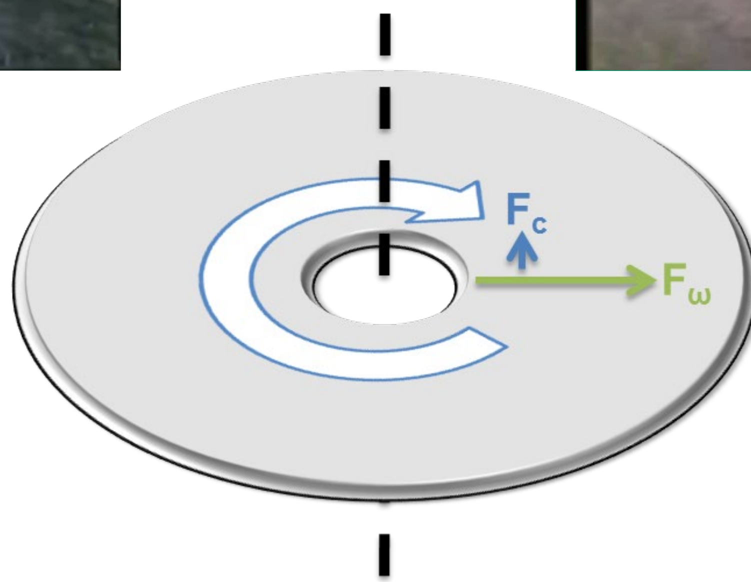
Centrifugal Microfluidics

Centrifugal Force



Coriolis Force

Centrifugal Microfluidics: Force Densities

Centrifugal (artificial gravity)

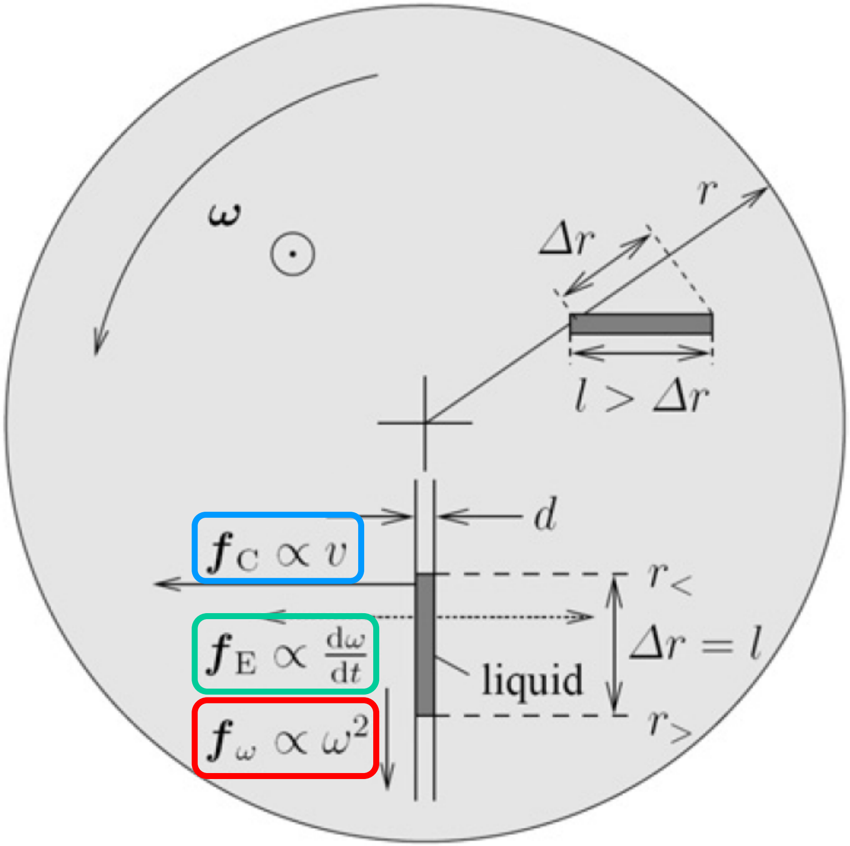
$f_\omega = \rho r \omega^2$

Euler (annular acceleration)

$f_\mathrm{E} = \rho r \dfrac{d\omega}{dt}$

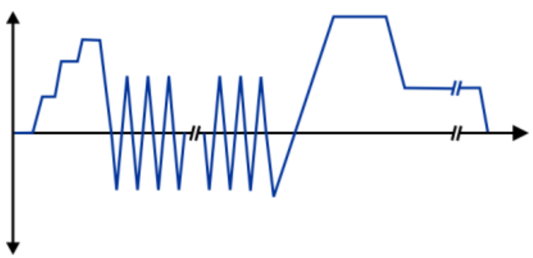
Coriolis (deflection of flows)

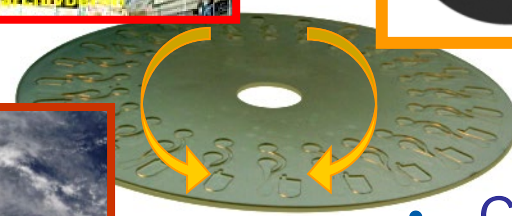
$f_\mathrm{C} = 2\rho\omega v$

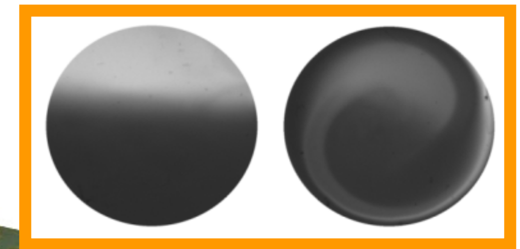


Centrifugal Microfluidics

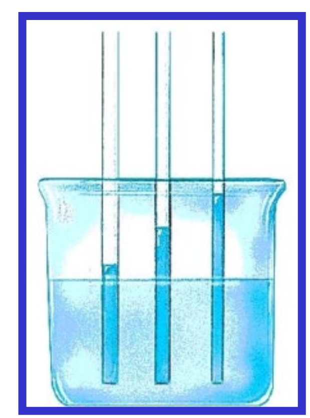
- Inertia
  - ➢ Centrifugal force $\propto r\omega^2$
  - ➢ Euler force $\propto d\omega/dt$
  - ➢ Coriolis force $\propto rv\omega$







- Capillary action

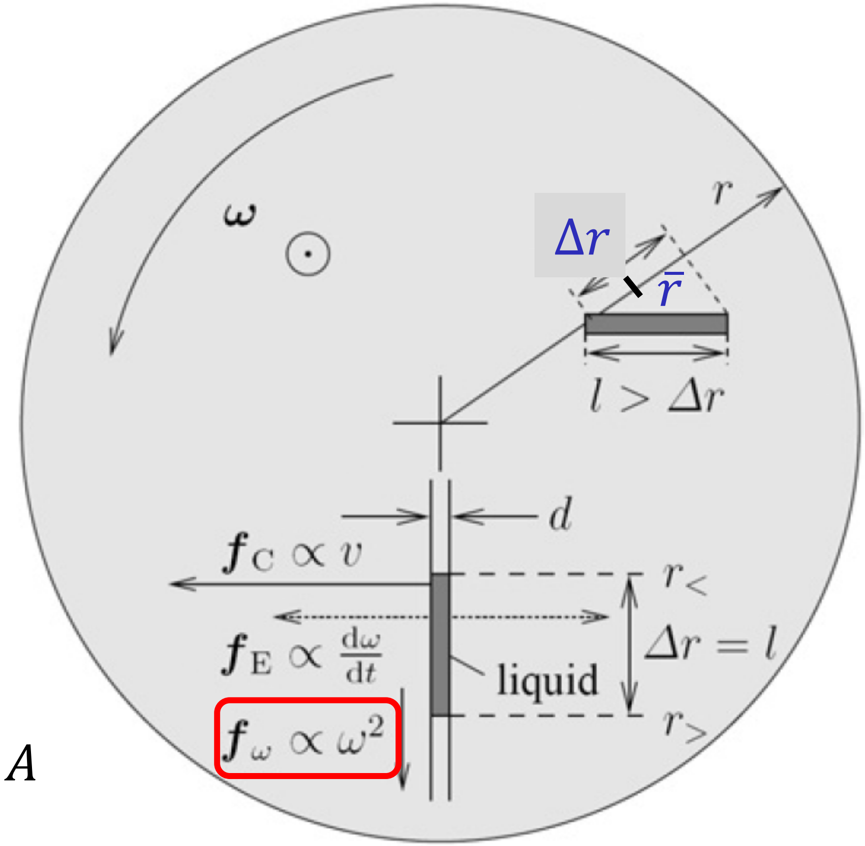


- Microstructured substrate

Centrifugal Microfluidics: Pressure Head

Channel segment of constant cross section A

Liquid segment:
- Density: ρ
- Radial extension: $\Delta r = r_{>} - r_{<}$
- Mean radial position: $\bar{r}$



Centrifugal force density

$f_\omega = \rho r \omega^2$

Centrifugal force on segment:

$$F_\omega = \rho\omega^2 A \int_{r_<}^{r_>} r dr = \frac{1}{2} A\rho\omega^2 (r_>^2 - r_<^2)$$

$$F_\omega = \rho A \cdot \underbrace{\frac{1}{2}(r_< + r_>)}_{\bar{r}} \cdot \underbrace{(r_> - r_<)}_{\Delta r} \cdot \omega^2 = p \cdot A$$

Equivalent pressure head: $p_\omega = \rho \cdot \bar{r} \cdot \Delta r \cdot \omega^2$

Centrifugal Microfluidics: Flow Profile

Pressure-driven flow through tube:
(Hagen-Poiseuille)

- Lateral coordinate of tube: x
- Lateral radius of tube: x_0

$$v_z(r) = \frac{\Delta p}{4\eta l}(x_0^2 - x^2)$$

Equivalent pressure head of segment

- Mean radial position $\bar{r}$
- Radial extension $\Delta r = l$

$$p_\omega = \rho \cdot \bar{r} \cdot \Delta r \cdot \omega^2$$

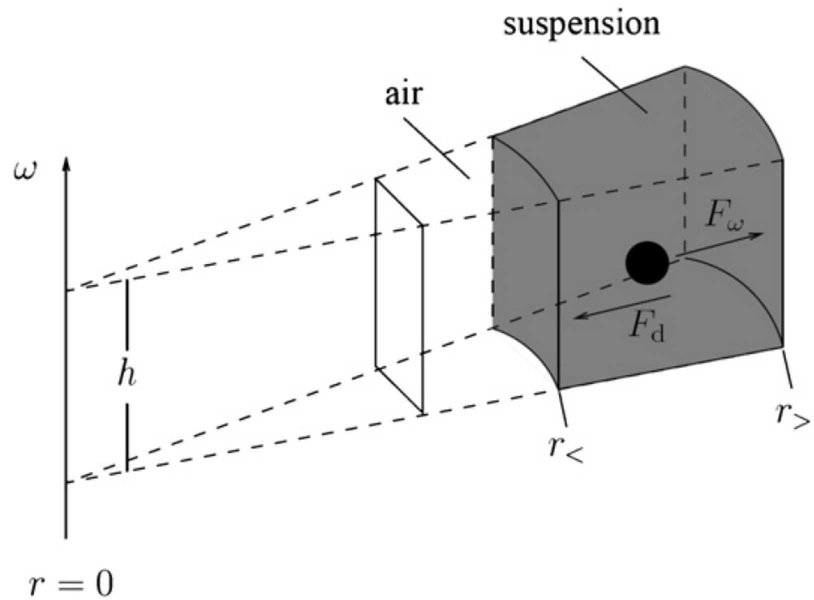
Maximum velocity at centre ($x = 0$)

$$v_{\max} = \frac{\rho \cdot \bar{r} \cdot \omega^2}{4\eta} x_0^2$$

Parabolic velocity profile

$$v_z(r) = v_{\max} - \frac{\rho \cdot \bar{r} \cdot \omega^2}{4\eta} x^2$$

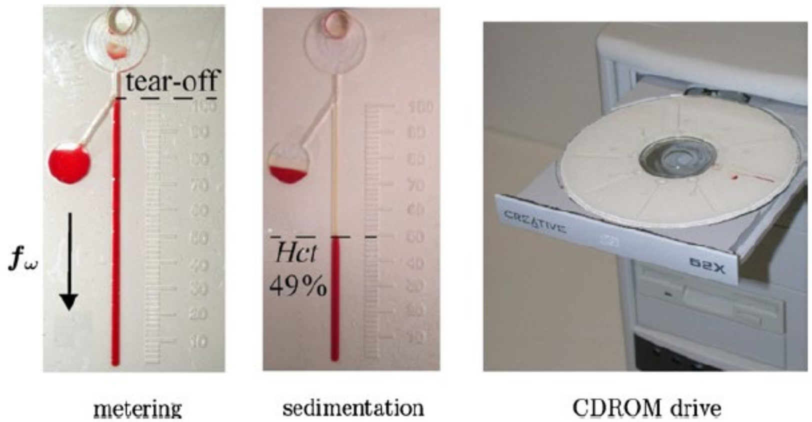
Centrifugal Microfluidics: Sedimentation

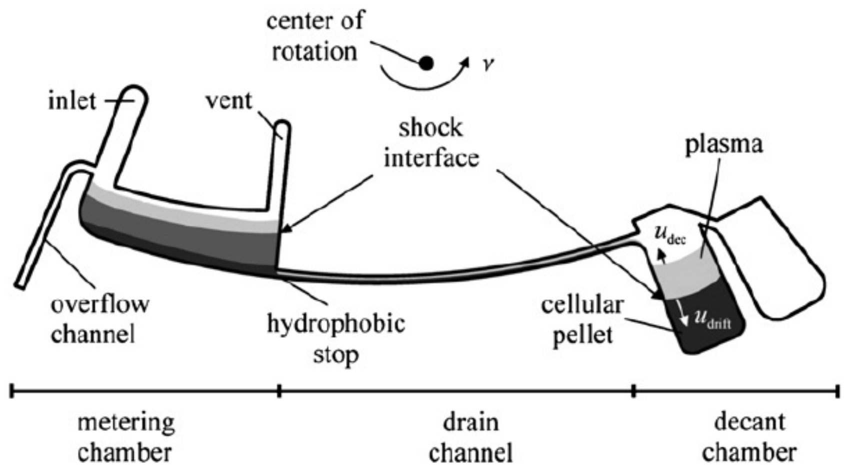


- Object suspended in stationary liquid phase
  - Sphere of radius x_0
  - Radial location r
  - Mass difference Δm with respect to suspending medium
- Constant drift velocity v_{drift}
  - Buoyancy force: $F_\omega = \Delta m r \omega^2$
  - Stokes drag: $F_{\mathrm{Stokes}} = 6\pi\eta x_0 v$

$$v_{\mathrm{drift}} = \frac{\Delta m r \omega^2}{6\pi\eta x_0}$$

Centrifugal Microfluidics: Shock Interface





$$r_*(t) = r_< e^{\mathcal{S}_0 \omega^2 t}$$

- Whole blood
- Suspended cells
  - Primarily RBCs
- Emergence of plasma interface $r_<(t)$
- Sedimentation coefficient

$$\mathcal{S} = \frac{v_r}{\omega^2 r} = \frac{m}{\bar{f}} \left(1 - \frac{\varrho_{\text{liquid}}}{\varrho_{\text{part}}}\right)$$

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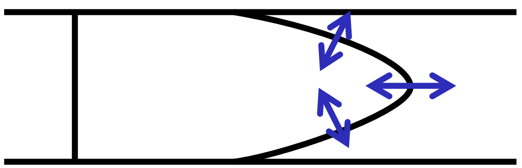
3.2.8. Taylor Dispersion



- Axial dispersion of solute in laminar flow
  - Dispersion of drugs in blood flow
- Situation
  - Steady state flow
  - Round tube
- Hypothetical absence of diffusion
  - Solute follows flow profile
- Molecular diffusion
  - Counteracts dispersion
  - Axial spreading at $\sqrt{D_{\text{eff}} t}$
  - Radial diffusion exchanges solute molecules between layers
- MF example
  - $v = 1$ mm s^{-1}, $r_0 = 100$ µm, $D = 3 \times 10^{-9}$ m^2 s^{-1}
  - Second term prevails over unity
  - Effective constant for axial diffusion $\sim D(1 + c\, D^{-2})$

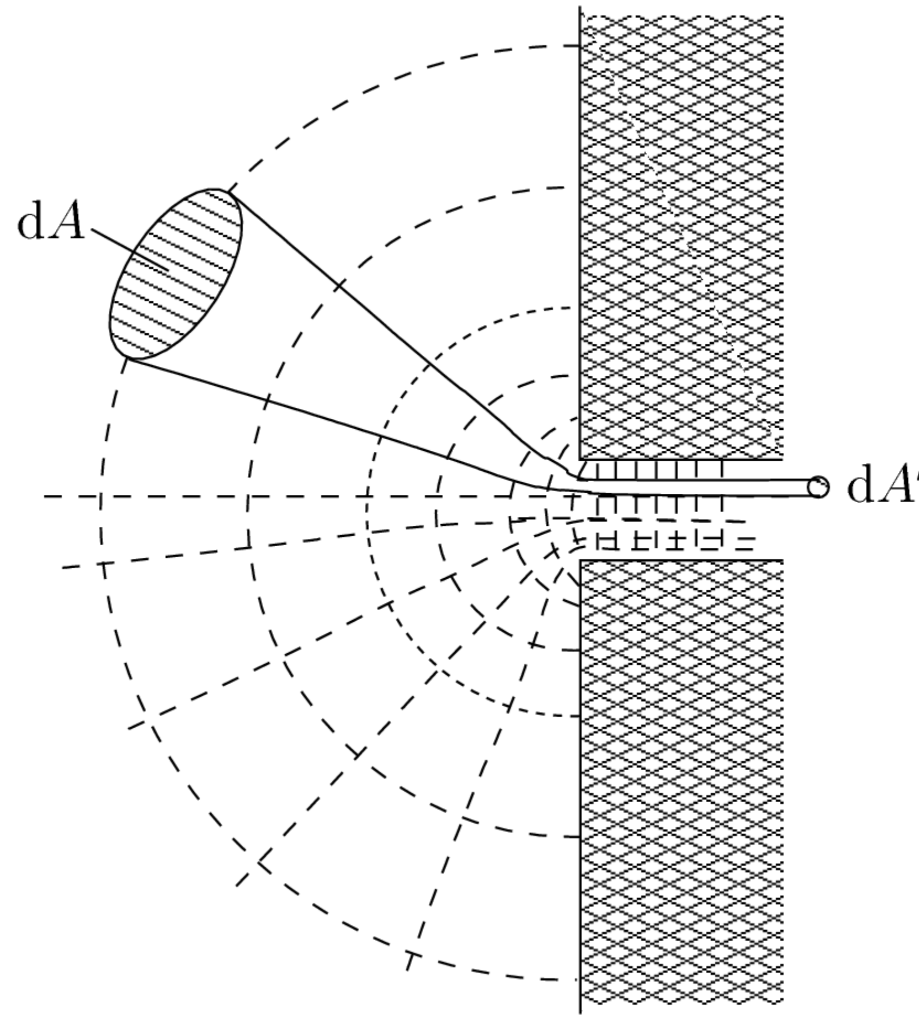
$$\frac{\partial \bar{c}}{\partial t} + \bar{v}\frac{\partial \bar{c}}{\partial z} = \mathcal{D}\frac{\partial^2 \bar{c}}{\partial z^2}$$

$$\mathcal{D} = D\left(1 + \frac{1}{48}\frac{\bar{v}^2 r_0^2}{D^2}\right)$$

3.2.8. Hydrodynamic Focusing

- Situation
  - Microscopic tip at end of capillary
  - Sucking in liquid from larger vessel
  - Laminar regime

- Full solid angle projected onto tiny orifice cross section



3.2.8. Hydrodynamic Focusing

- Example:
  - Ink dispenser near orifice of capillary
  - Vertical position within capillary adjusts to transversal shift of dispenser

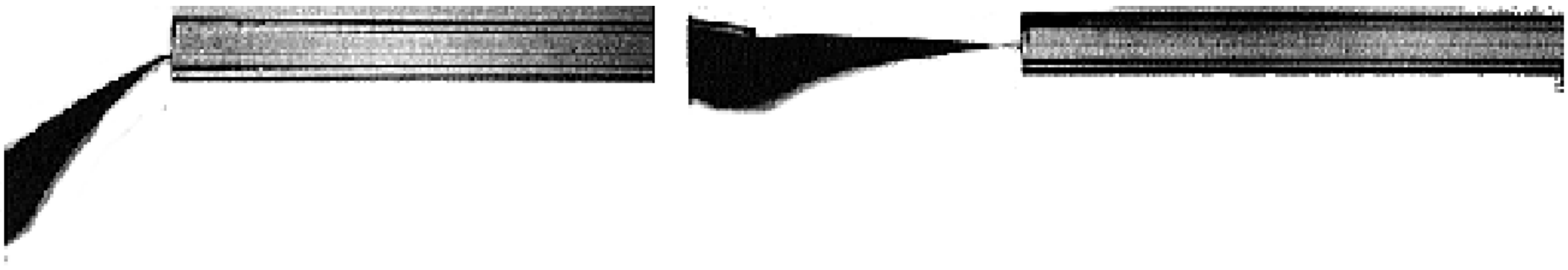


Fig. 3.12. By hydrodynamic focusing, a thread of ink is drawn into the capillary. Depending on the position of the ink dispenser, the vertical position of the thread within the capillary shifts

3.2.8. Reversed Experiment

- Fluid plug expelled from orifice of capillary into larger tank
- Small velocity
  - Laminar
- High velocity
  - Turbulent

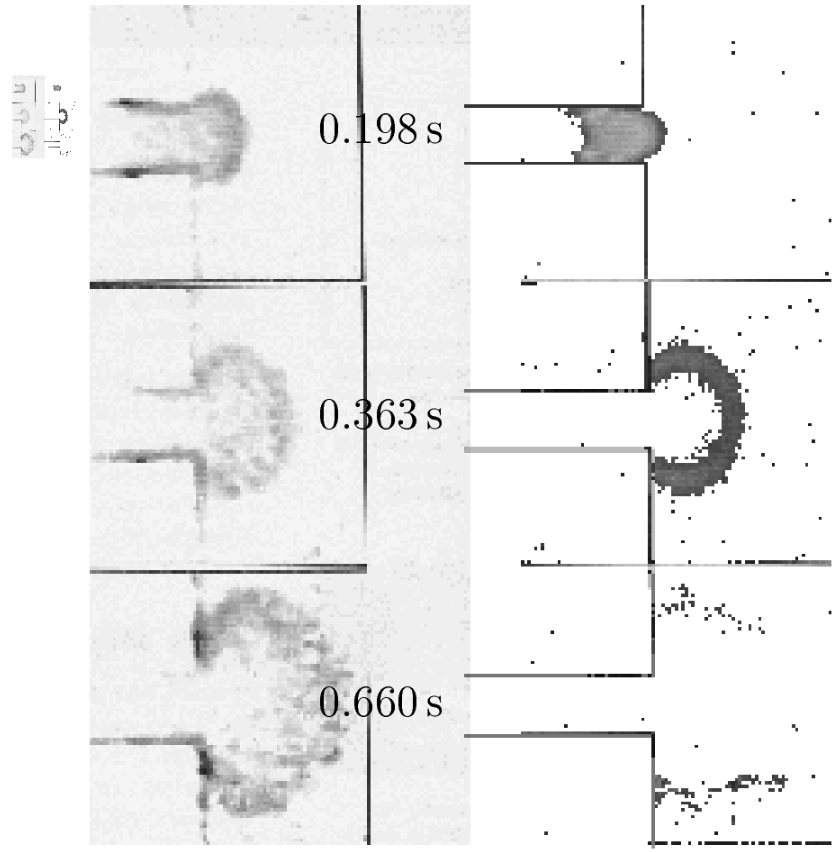


Fig. 3.13. Fluid plug exiting a capillary in the turbulent (left) and laminar regime (right). As long as laminar flow conditions are preserved, a reversal of hydrodynamic focusing of Fig. 3.12 is observed which later resolves into turbulence

3.2.8. Application to Cytometry and Mixing

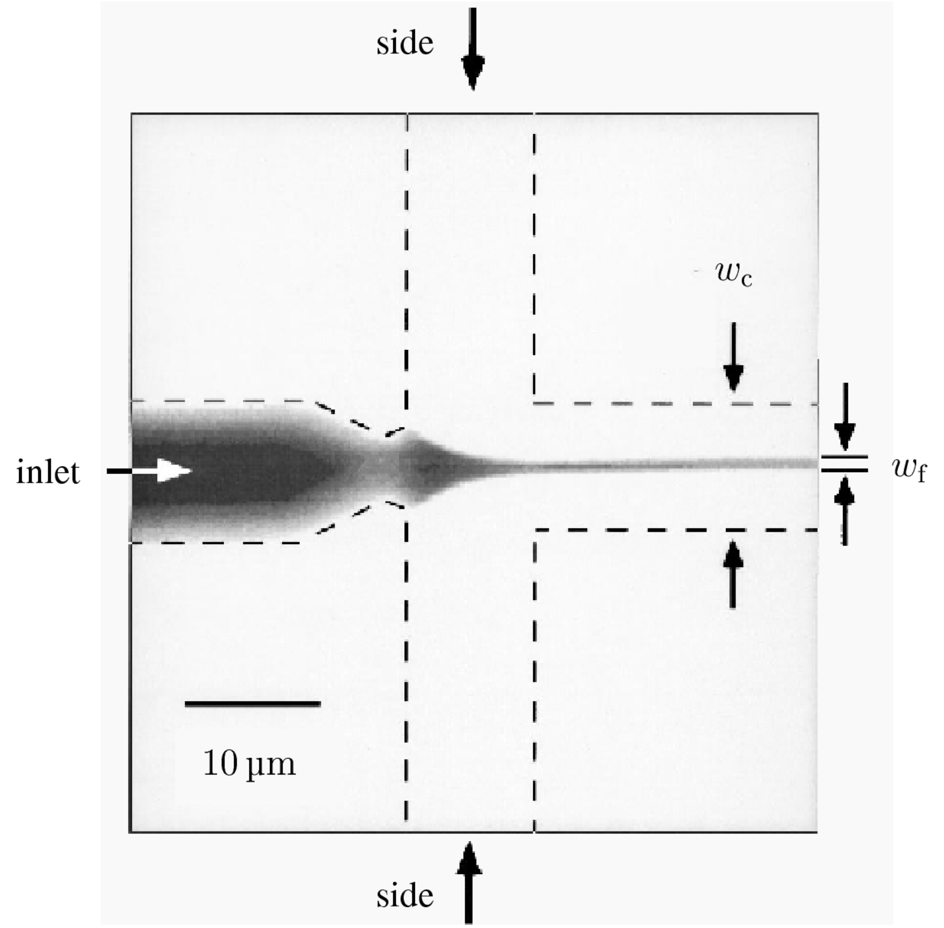


Fig. 3.14. Hydrodynamic focusing used for mixing inlet and side channel streams on the millisecond scale. A side channel pressure of 344.6 hPa and a ratio of 1.1 between side and inlet channel pressure are applied. The edges of the channels are outlined with a dashed line for reference. The variables w_c and w_f describe the width of the channel (10 µm) and the width of the focused inlet stream

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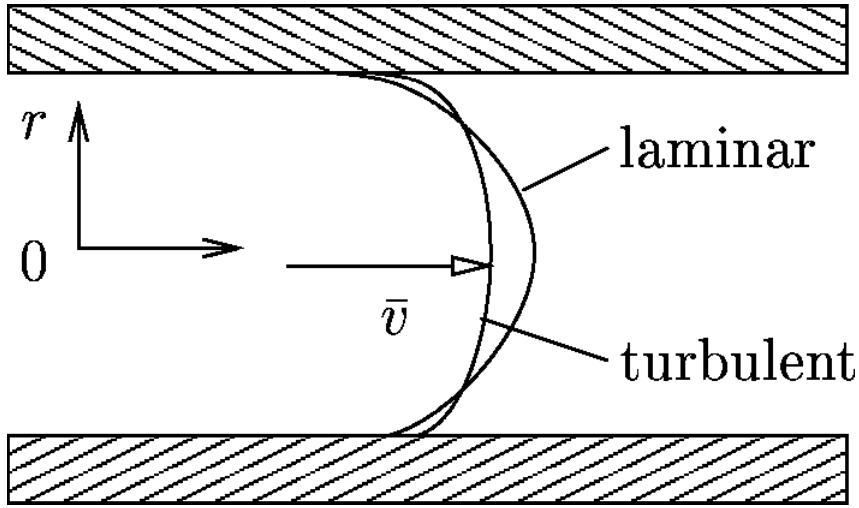
3.2.9. Turbulent Flows



- Turbulent flow in tube for $Re > Re_{\text{crit}}$
- Turbulent profile
  - Velocity vectors unpredictably oscillating in time
  - Time-averaged profile
  - Much flatter profile than laminar flow
  - Tendency for flattening grows with Re

3.2.9. Turbulent Flows

optional

- Throughput according to Blasius (1883-1970)
  - Approximations well above $3Re^*$

Laminar

$$I_V = \frac{\pi}{8\eta}\frac{\Delta p}{l} r_0^4$$

$$I_V^{\text{turb}} = \frac{\mathrm{d}V}{\mathrm{d}t} = A\bar{v}_{\text{turb}} = 4.71\pi \left[\left(\frac{\Delta p}{l} \right)^4 \frac{r_0^{19}}{\eta \varrho^3} \right]^{1/7}$$

$$\propto \left(\frac{\Delta p}{l} \right)^{0.57} r_0^{2.71}$$

- Mean velocity

$$\bar{v} = \frac{\Delta p}{8\eta l} r_0^2$$

$$\bar{v}_{\text{turb}} = 4.71 \left[\left(\frac{\Delta p}{l} \right)^4 \frac{r_0^5}{\eta \varrho^3} \right]^{1/7}$$

$$\propto \left(\frac{\Delta p}{l} \right)^{0.57} r_0^{0.71}$$

3.2.9. Scaling of Mean Velocity

optional

Turbulent		Laminar	
$\bar{v}_{\mathrm{turb}} \propto \left(\frac{\Delta p}{l}\right)^{4/7}$	and	$\bar{v}_{\mathrm{lam}} \propto \frac{\Delta p}{l}$	Pressure gradient
$\bar{v}_{\mathrm{turb}} \propto r_0^{5/7}$	and	$\bar{v}_{\mathrm{lam}} \propto r_0^2$	Radius
$\bar{v}_{\mathrm{turb}}(\varrho) \propto \varrho^{-3/7}$	and	$\bar{v}_{\mathrm{lam}}(\varrho) = \mathrm{const.}$	Density
$\bar{v}_{\mathrm{turb}}(\eta) \propto \eta^{-1/7}$	and	$\bar{v}_{\mathrm{lam}} \propto \eta^{-1}$	Viscosity

- Same pressure gradient applied to tube
  - Smaller turbulent flow velocity
- Turbulent velocity varies with density ρ
  - Flow energy dissipated by turbulent mixing
- Laminar flow
  - Viscous forces between smoothly sliding layers
- Turbulent regime
  - Enhanced flow resistance

3.2.9. Entrance Effects

optional

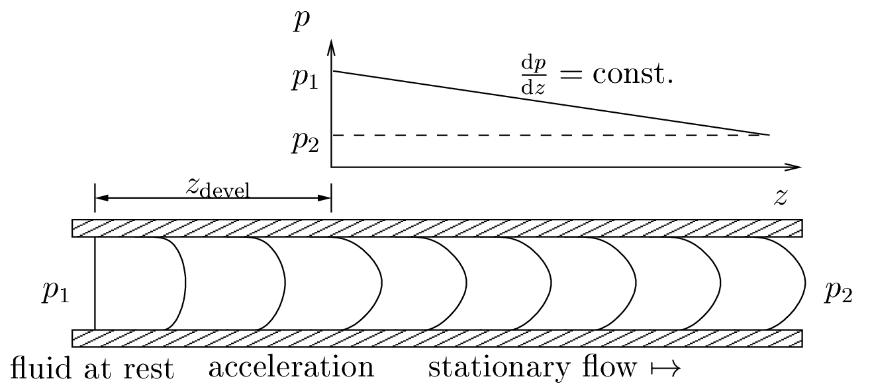


Fig. 3.17. Evolution of the velocity profile $v(r)$ in the entrance region of a tube for a constant pressure gradient after laminar flow regime has been entered

- Laminar

$$z_{devel} \simeq 0.12 r_0 Re$$

- Turbulent

$$z_{devel} \simeq 8.8 r_0 Re^{1/6}$$

- Microfluidic systems
  - $Re \sim 1$ and $r_0 = 100\ \mu m$

$z_{devel} = 10\ \mu m$ (laminar)

3.2.9. Friction Losses

optional

- Surface roughness
- Local eddy formation
- Darcy-Weissbach relation
  - Pressure loss
  - Flow velocity
  - Friction factor

$$\Delta p_\epsilon = f_\epsilon \frac{\varrho l v^2}{2d}$$

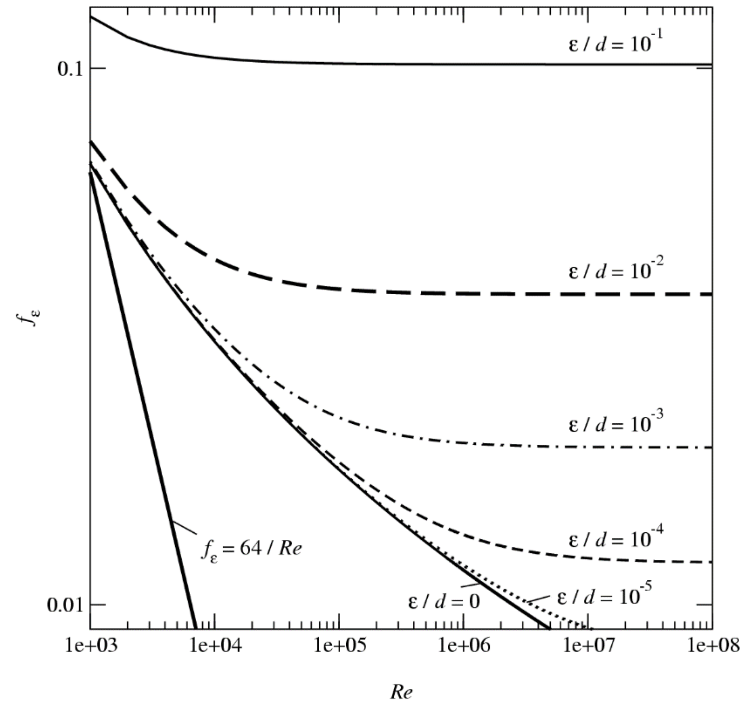


Fig. 3.18. Moody chart expressing the Darcy friction factor f_ϵ as a function of the Reynolds number Re as obtained from (??). The curves are parametrized by the relative surface roughness ϵ/d. The vertical line $Re = 2300$ separates the laminar from the turbulent regime

$$f_\epsilon = f_\epsilon(Re, \frac{\epsilon}{d}, A) = \frac{\tau_{\mathrm{wall}}}{\frac{1}{2}\varrho v^2}$$

f_ε = const. for smooth tube and laminar conditions

3.2.9. Roughness-Viscosity Model

optional

- Surface roughness induces turbulence in boundary layer
  - Surface roughness height ε
- Roughness viscosity

$$\eta_\epsilon(r) = \eta A_\epsilon Re_\epsilon \frac{r}{\epsilon}\left[1 - \exp\left(-\frac{Re_\epsilon}{Re}\frac{r}{\epsilon}\right)\right]^2$$

  - Surface roughness Reynolds number

$$Re_\epsilon = \frac{\epsilon v_\epsilon}{v} \simeq \left(\frac{\partial v}{\partial r}\right)_{r=r_0} \frac{\epsilon^2}{v}$$

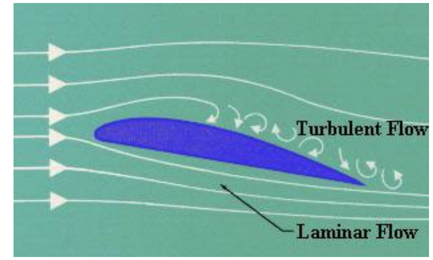
  - Empirical factor

$$A_\epsilon = 0.1306\left(\frac{r_0}{\epsilon}\right)^{0.3693} \exp\left[Re\left(6 \times 10^{-5}\frac{r_0}{\epsilon} - 0.0029\right)\right]$$

Summary I

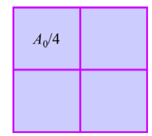
(Critical) Reynolds number

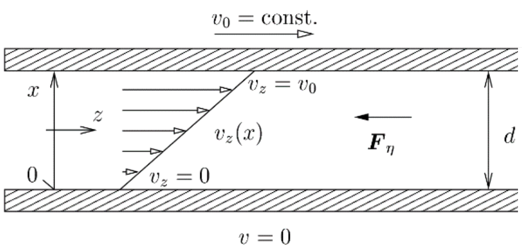
$$Re = \frac{\varrho_\infty \tilde{v} \tilde{l}}{\eta} = \frac{\tilde{v}\tilde{l}}{\upsilon}$$

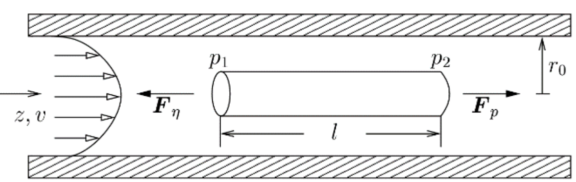


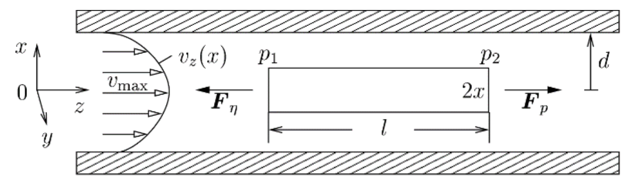
Shear-driven flow

Pressure-driven flow









$$v_z(x) = \frac{x}{d} v_0$$

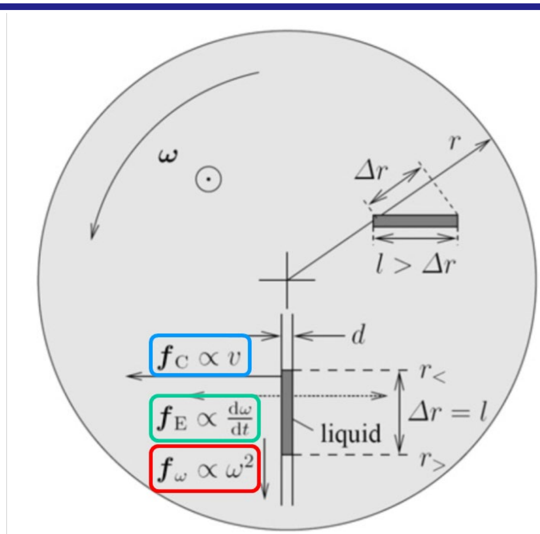
$$I_V = \frac{\pi}{8\eta} \frac{\Delta p}{l} r_0^4$$

$$v_{\max} = \frac{\Delta p}{2\eta l} d^2$$

$$v_z(r) = \frac{\Delta p}{4\eta l}(r_0^2 - r^2) \quad v_{\max} = \frac{\Delta p}{4\eta l} r_0^2$$

$$\frac{I_V}{y} = 2\int_0^d v(x)\,\mathrm{d}x = \frac{h^3}{12\eta}\frac{\mathrm{d}p}{\mathrm{d}z}$$

Summary II



Rotationally induced force densities

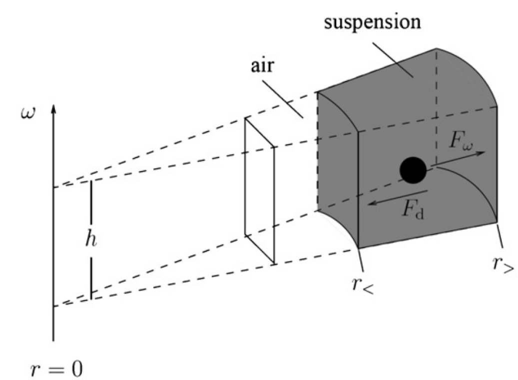
Centrifugal (artificial gravity): $f_\omega = \rho r \omega^2$

Euler (annular acceleration): $f_E = \rho r \frac{d\omega}{dt}$

Coriolis (deflection of flows): $f_C = 2\rho\omega v$

Particle sedimentation

$$v_{\mathrm{drift}} = \frac{\Delta m r \omega^2}{6\pi\eta x_0}$$



Pressure head

$$p_\omega = \rho \cdot \bar{r} \cdot \Delta r \cdot \omega^2$$

Flow profile

$$v_z(r) = \frac{\Delta p}{4\eta l}(x_0^2 - x^2)$$